2024

MATHEMATICS — HONOURS

Paper : DSE-A-2.1, DSE-A-2.2 and DSE-A-2.3

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

DSE-A-2.1

(Differential Geometry)

Full Marks : 65

[The symbols have their usual meaning.]

1. Answer the following multiple choice questions. For each question, **1 mark** for choosing correct option and **1 mark** for correct justification : 2×10

(a) Krönecker delta δ^i_j is

(i) a covariant tensor of rank two (ii) a contravariant tensor of rank two

(iii) a mixed tensor of rank two (iv) not a tensor.

(b) The value of g_{23} and g^{33} where the line element in a Riemannian space V_3 is given by $ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(dx^3)^2 - 3(dx^1)(dx^2) - 3(dx^2)(dx^1) + 2(dx^2)(dx^3) + 2(dx^3)(dx^2)$ are equal to

(i) $2, \frac{3}{2}$ (ii) $-3, \frac{3}{2}$

(iii) $2, \frac{5}{2}$ (iv) $-3, -\frac{5}{2}$.

(c) In a cartesian co-ordinate system E^3 the geodesics are

(i) circle (ii) parabola

(iii) straight line (iv) ellipse.

(d) The number of independent components of the fundamental metric tensor g_{ij} is

(i) exactly $\frac{n(n+1)}{2}$ (ii) at least $\frac{n(n+1)}{2}$

(iii) at most $\frac{n(n+1)}{2}$ (iv) none of these.

(e) If A^{ijk} is a skew-symmetric tensor, then

(i) $A^{ijk}\begin{Bmatrix} i \\ j\,l \end{Bmatrix}=0$ (ii) $A^{ijk}\begin{Bmatrix} j \\ i\,l \end{Bmatrix}=0$

(iii) $A^{ijk}\begin{Bmatrix} i \\ k\,j \end{Bmatrix}=0$ (iv) $A^{ijk}\begin{Bmatrix} l \\ i\,j \end{Bmatrix}=0$.

(f) For an Einstein space, Ricci tensor can be represented by

(i) $R_{ij}=\dfrac{R}{n^2}g_{ij}$ (ii) $R_{ij}=\dfrac{R}{n}g_{ij}$

(iii) $R_{ij}=\dfrac{R}{n(n+1)}g_{ij}$ (iv) $R_{ij}=0$.

(g) If A_i is the associate of a contravariant vector A^i and $\dfrac{\delta A^i}{\delta t}=0$, then $\dfrac{\delta A_i}{\delta t}$ is equal to

(i) 0 (ii) 1

(iii) −1 (iv) none of these.

(h) In the Riemannian space V_4 with line element $ds^2=-\left(dx^1\right)^2-\left(dx^2\right)^2-\left(dx^3\right)^2+c^2\left(dx^4\right)^2$, the covariant vector $\left(-1,0,0,\dfrac{1}{c}\right)$ is

(i) a null vector (ii) not a unit vector

(iii) a unit vector (iv) none of these.

(i) If a surface is isometric with Euclidean plane E_2, then

(i) $K>0$ (ii) $K<0$

(iii) $K=0$ (iv) none of these.

(where K is the Gaussian curvature of the surface).

(j) A surface M is a minimal surface if

(i) $k_1k_2=0$ (ii) $k_1+k_2=0$

(iii) $K=0$ (iv) $k_1=k_2$.

(where k_1, k_2 are principal curvatures and K is the Gaussian curvature).

Unit - 1

Answer ***any one*** question. 5×1

2. Find the Christoffel symbols $\left\{ \begin{matrix} 2 \\ 1\ 2 \end{matrix} \right\}$ and $\left\{ \begin{matrix} 2 \\ 2\ 3 \end{matrix} \right\}$ in a 3-dimensional Riemannian space, in which the line element is given by $ds^2 = \left(dx^1\right)^2 + \left(x^1\right)^2\left(dx^2\right)^2 + \left(x^1 \sin x^2\right)^2\left(dx^3\right)^2$.

3. Prove that $\left\{ \begin{matrix} i \\ i\ j \end{matrix} \right\} = \frac{\partial}{\partial x^j} \log\sqrt{g}$, where $g = |g_{ij}| > 0$.

Unit - 2

Answer ***any four*** questions. 5×4

4. Show that $\frac{\delta^2 t^i}{\delta s^2} = \frac{d\kappa}{ds} n^i + \kappa\left(-\kappa t^i + \tau b^i\right)$.

5. Find the first fundamental form for the surface

$$x^1 = a\cos u^1 \sin u^2,$$
$$x^2 = a\sin u^1 \sin u^2,$$
$$x^3 = c\cos u^2,$$

where a and c are constants.

6. Find the equation for the principal curvatures of the surface for the right helicoid (u cos v, u sin v, cv).

7. Find the curvature and torsion of the curve $\vec{r} = \left(\tan^{-1} s\right)\hat{i} + \frac{1}{\sqrt{2}}\log\left(s^2+1\right)\hat{j} + \left(s - \tan^{-1} s\right)\hat{k}$.

8. Prove that the relation $K = \det\left(b^{\alpha}_{\beta}\right)$ holds for $a = \det\left(a_{\alpha\beta}\right), b = \det\left(b_{\alpha\beta}\right)$, and $K = \frac{b}{a}$.

9. Prove that there are points on the cubic $x = at^3 + b$, $y = 3ct^2 + 3dt$, $z = 3et + f$ such that the osculating plane passes through the origin and the points lie on the plane $3cex + afy = 0$.

10. Show that the parametric curves are asymptotic lines to the surfaces

$$x^1 = u^1 \cos u^2; x^2 = u^1 \sin u^2; x^3 = cu^2.$$

Unit - 3

Answer ***any four*** questions.

11. Find the Gaussian curvature of the surface $(x^1)^2 - (x^2)^2 = 4x^3$, at the point (2, 0, 1). 5

12. Show that the conditions that the curve u^1-curve and u^2-curve be geodesics are $\left\{ {2 \atop 1\ 1} \right\} = 0$ and $\left\{ {1 \atop 2\ 2} \right\} = 0$ respectively. 5

13. Prove that the Gaussian curvature is an invariant. 5

14. Find the differential equations of the geodesic for the metric $ds^2 = \left(dx^1\right)^2 + \left\{\left(x^1\right)^2 + c^2\right\}\left(dx^2\right)^2$. 5

15. When a surface is called developable? Determine whether the surface with the metric

$$ds^2 = (u^2)^2(du^1)^2 + (u^1)^2(du^2)^2$$

is developable or not. 1+4

16. Prove that the normal curvatures in the directions of the coordinate curves are $\frac{b_{11}}{a_{11}}$ and $\frac{b_{22}}{a_{22}}$, respectively. 5

17. If the lines of curvature are not indeterminant at a given point P on the surface and if α is the angle between a given direction and a principal direction at P, then the normal curvature is given by, $\kappa_{(n)} = \kappa_1 \cos^2\alpha + \kappa_2 \sin^2\alpha$. 5

DSE-A-2.2

(Mathematical Modelling)

Full Marks : 65

Group - A

1. Answer *all* questions : 2×10

(Each question has four alternatives of which one is correct. Choose the correct one with proper justification. **One mark** is for correct answer and **one mark** is for proper justification.)

(a) Power series solution of $x^3(x-1)^2(x-2)\dfrac{d^2y}{dx^2}+2(x-3)^2\dfrac{dy}{dx}+4(x-5)y=0$ exists at

(i) $x=0$ (ii) $x=1$

(iii) $\forall x \in R$ (iv) $x=3$.

(b) $x\{J_{n-1}(x)+J_{n+1}(x)\}=$

(i) $2J_n(x)$ (ii) $2J_n'(x)$

(iii) $2_n J_n(x)$ (iv) $J_n(x)$.

(c) If $L\{f(t)\}=F(s)$, then $L\{f'(t)\}$ is

(i) $sF(s)-F(0)$ (ii) $sF(s)-f(0)$

(iii) $sF(s)-f'(0)$ (iv) $F(s)-F(0)$.

(d) Laplace transform of the function $2\sqrt{\dfrac{t}{\pi}}$ is

(i) $\dfrac{2}{\sqrt{\pi p}}$ (ii) $\dfrac{1}{\sqrt{p^3}}$

(iii) $\dfrac{1}{\sqrt{\pi p^2}}$ (iv) $\dfrac{2}{\sqrt{p}}$.

(e) Using Linear Congruence method $x_{n+1}=5x_n+7 \bmod (101)$ with seed $x_0=144$, then 3rd random number generated is

(i) 37 (ii) 20

(iii) 31 (iv) 6.

(f) A duplicating machine is operated by an office assistant. If jobs arrive at a rate of 5 per hour and the time to complete each job varies according to an exponential distribution with mean 6 min. Then the probability that the assistant is idle will be

(i) $\frac{1}{2}$ (ii) $\frac{1}{3}$

(iii) $\frac{1}{4}$ (iv) $\frac{2}{3}$.

(g) Cars arrive at a car wash centre following Poisson distribution. If the arrival rate is 20 per hour and if one car takes exactly 2 minutes, then average waiting time is

(i) 4 mins (ii) 3 mins

(iii) 2 mins (iv) 1 min.

(h) Number of extreme points on the convex set $\left\{(x, y): x^2 + \frac{y^2}{4} \le 1,\ x \ge 0,\ y \ge 0\right\}$ is

(i) 2 (ii) 3

(iii) 4 (iv) infinite.

(i) The value of $\int_{-1}^{+1} P_n(x)\, dx$ for any non-negative integer n lies in which following set?

(i) $\{-1,\ 1\}$ (ii) $\{\ 0\ \}$

(iii) $\{0, 2\}$ (iv) $\{-2,\ 0\}$.

(j) If $x^2 = a\ P_0(x) + b\ P_2(x)$, then

(i) $a = \frac{2}{3},\ b = \frac{1}{3}$ (ii) $a = 2,\ b = 1$

(iii) $a = \frac{1}{3},\ b = \frac{2}{3}$ (iv) $a = b = \frac{2}{3}$.

Group - B

Unit - I

2. Answer ***any two*** questions :

(a) (i) If $y = \sum a_m x^m$ is a solution of $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 3y = 0$, then show that

$$\frac{a_m}{a_{m+2}} = -\frac{(m+1)(m+2)}{m+3}.$$

(ii) Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = x\cos x$ with $y(0) = 0 = \frac{dy}{dx}(0)$ by Laplace transform. 5+5

(b) (i) Show that $J^2_{-\frac{1}{2}}(x) + J^2_{+\frac{1}{2}}(x) = \frac{2}{\pi x}$, where $J_n(x)$ is Bessel function of first kind of order n.

(ii) Prove that $J_n'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$. 5+5

(c) (i) If $P_n(x)$ denotes Legendre polynomial of degree n, then show that

$$P_n(x) = \int_0^{\pi}\left(x \pm \sqrt{x^2 - 1}\cos\phi\right)^n d\phi.$$

(ii) Prove that $(1 - x^2)P_n'(x) = n(P_{n-1}(x) - xP_n(x))$. 5+5

(d) (i) Find the inverse Laplace transform of $\log\left(1 - \frac{1}{p^2}\right)$.

(ii) Using Laplace transform, show that $\int_0^{\infty}\frac{e^{-t}\sin t}{t}dt = \frac{\pi}{4}$. 5+5

Unit - II

3. Answer *any five* questions : 5×5

(a) Solve the following L.P.P. by graphical method :

$$\begin{aligned} \text{Maximize } Z &= -x_1 + 3x_2 \\ \text{subject to } \quad & x_1 - x_2 \geqslant -1 \\ & -0.5x_1 + 1.5x_2 \leqslant 3 \\ \text{and } \quad & x_1, x_2 \geqslant 0 \end{aligned}$$

(b) Solve the following L.P.P. by algebraic method :

$$\begin{aligned} Z_{\max} &= 10x_1 + 15x_2 \\ \text{subject to } \quad & x_1 + x_2 \geqslant 2 \\ & 3x_1 + 2x_2 \leqslant 6 \\ \text{and } \quad & x_1, x_2 \geqslant 0 \end{aligned}$$

(c) A departmental store has only one cashier. If customers arrive at a rate of 20 customers per hour and the cashier handles 24 customers per hour. Assuming single channel queuing model, find out :

(i) average customer in the system

(ii) average customer in the queue

(iii) average time a customer spends in queue.

(d) For a M/M/1 : ∞/FIFO queuing model in steady-state case, show that expected number in the queue is $\mu/(\mu-\lambda)$ assuming queue is not empty.

(e) Describe Middle square method to generate random numbers. Using this method, find six random numbers strating from 1234.

(f) Given the L.P.P. $Z_{max} = 60x + 50y$

with $x + 2y \leqslant 40$

$3x + 2y \leqslant 60$

and $x, y \geqslant 0$

Determine ranges for discrete changes of 'b' when changed one at a time, to maintain the optimality of the given L.P.P.

(g) A photography studio uses an expensive grade of developing fluid when printing special colour portraits. Since the developing fluid cannot be stored for long periods, it is important to keep on hand only as much as is needed to fill anticipated demand. In the past few months, however, demand for the product has been fluctuating. The owner has decided to simulate the demand for this service.

A study of the studio's appointment book resulted in the following frequency distribution :

Daily demand	:	0	1	2	3	4	5
Number of days	:	10	20	40	20	6	4

The data was taken for a 100-day period during which no more than five special prints were requested on any given day. Using the data given above, generate a ten-day sequence of demand values. Use the following random numbers :

67 63 39 55 29 78 70 06 78 76

(h) During morning rush hour, motor vehicles pass a section of road at rate 32 per minute. Sam regularly crosses over this section of road, during this morning rush hour. It takes her 1/4th of a minute to cross over the road. What is the probability that at least 6 vehicles pass this section of road while Sam is crossing? Also find the probability that more than 10 vehicles pass this section of road while Sam is crossing.

DSE-A-2.3

(Fluid Statics and Elementary Fluid Dynamics)

Full Marks : 65

Symbols have their usual meanings.

1. Answer all questions with proper explanation / justification (**one mark** for correct answer and **one mark** for justification) : 2×10

(a) Effect of viscosity is neglected in

(i) real fluid (ii) Newtonian fluid

(iii) ideal fluid (iv) non-Newtonian fluid.

(b) If dp denotes the increase of pressure corresponding to the compression of volume dv of a gas of volume v, then the compressibility is

(i) $-\frac{1}{v}\frac{dv}{dp}$ (ii) $-\frac{1}{v}\frac{dp}{dv}$

(iii) $-v\frac{dv}{dp}$ (iv) $\frac{dp}{dv}$.

(c) The depth of C.P. of a plain triangular lamina of height $2h$ immersed in a fluid with its plane vertical, vertex on the free surface and base parallel to the free surface is

(i) $\frac{h}{2}$ (ii) $\frac{3h}{2}$

(iii) $\frac{h}{4}$ (iv) $\frac{3h}{4}$.

(d) A floating body is said to be in a state of stable equilibrium

(i) when its metacentric height is zero

(ii) when metacentre is above the centre of gravity

(iii) when metacentre is below the centre of gravity

(iv) only when its centre of gravity is below its centre of buoyancy.

(e) Isothermal process is characterized by

(i) $\frac{p}{\rho}$ = constant (ii) pT = constant

(iii) $\frac{p}{v}$ = constant (iv) $p\rho^{\gamma}$ = constant

(f) If p_1 and p_2 are the pressures at the points of depths h_1 and h_2 respectively in a homogeneous fluid at rest, then

(i) $\frac{p_2}{p_1} \propto \frac{h_1}{h_2}$ (ii) $p_1 + p_2 \propto h_1 + h_2$

(iii) $p_1 - p_2 \propto h_1 - h_2$ (iv) $p_1 h_1 \propto p_2 h_2$

(g) Given steady, incompressible velocity distribution $V = 3x\hat{i} + Cy\hat{j} + 4z\hat{k}$. If the conservation of mass is satisfied, the value of C should be

(i) -3 (ii) 0

(iii) -7 (iv) -4

(h) If the components of velocity field are $u = yz + t$, $v = zx - t$, $w = xy$, then z component of acceleration at the point (2, 1, 3) at $t = 1$ sec is

(i) 14 (ii) 15

(iii) 16 (iv) 18

(i) The streamlines of a flow $u = x$, $v = -y$ is given by

(i) $xy = c_1, z = c_2$ (ii) $xz = c_1, y = c_2$

(iii) $yz = c_1, x = c_2$ (iv) None of these.

(j) The motion of a inviscid fluid under conservative forces, if once irrotational, is always

(i) rotational (ii) irrotational

(iii) laminar (iv) None of these.

Unit - 1

2. Answer ***any one*** question :

(a) Prove that in a fluid at rest under gravity, the pressure is the same at all points in the same horizontal plane. 5

(b) Three fluids, whose densities are in A.P. fill a semicircular tube whose bounding diameter is horizontal. Prove that the depth of one of the common surfaces is double that of the other. 5

Unit - 2

3. Answer ***any two*** questions :

(a) (i) Prove that the necessary condition for equilibrium of a fluid under the action of external force whose components are X, Y, Z (all per unit mass) along the axes is

$$X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = 0.$$

(ii) A triangular lamina of base 'a' and altitude 'h', is placed in water with its plane vertical and the side 'a' horizontal and at a depth 'K' below the surface of the water. Prove that the depth of the centre of pressure is

$$\frac{6K^2 - 4hK + h^2}{2(3K - h)}.$$ 5+5

(b) (i) A cone whose vertical angle is 2α, has the lowest generator horizontal and is filled with liquid. Prove that the resultant thrust on the curved surface is $\sqrt{(1 + 15\sin^2\alpha)}$ times the weight of the liquid.

(ii) A mass of fluid rests upon a plane subject to a central attractive force $\frac{\mu}{r^2}$ to a point situated at distance c from the plane of the opposite side to that on which the fluid is situated. If a be the radius of the free spherical surface of the fluid, show that the thrust on the plane is $\frac{\pi\rho\mu}{a}(a-c)^2$, where ρ is the density of the fluid. 5+5

(c) (i) A homogeneous circular cylinder of length h, radius a and specific gravity σ, floats in water. Prove that the position with the axis vertical is stable if $\frac{a^2}{h^2} < 2(1-\sigma)\sigma$.

(ii) Prove that, if a plane section of a body floating freely in a heavy homogeneous fluid at rest, cuts off a volume which remains constant for small rotational displacement of the plane, the axis about which the plane turns passes through the centroid of the section. 5+5

(d) (i) Derive the expressions for pressure and density in an isothermal atmosphere at a height z above the sea level, considering gravity to be constant.

(ii) Masses m, m' of two gases in which the ratios of the pressure to the density are respectively K, K', are mixed at the same temperature. Prove that the ratio of the pressure to the density in the mixture is $\frac{mK + m'K'}{m + m'}$. 5+5

Unit - 3

4. Answer ***any one*** question :

(a) (i) The velocity components for a two-dimensional flow system are given by $u = 2y + 3$, $v = 2t$. Find the displacement of a fluid particle in the Lagrangian system, where the initial value of (x, y) is $(1, 2)$.

(ii) What is meant by the convective derivative? Show that the acceleration $\vec{a} = (\vec{q}\cdot\vec{\nabla})\vec{q} + \frac{\partial\vec{q}}{\partial t}$, where $\vec{q}$ is the velocity vector of the fluid particle. 5+(1+4)

(b) (i) If $\lambda = \frac{\partial u}{\partial t} - v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + w\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$, and μ, ν are two similar expressions, then prove that $\lambda dx + \mu dy + \nu dz$ is a perfect differential, if the external forces are conservative and the density is a constant; u, v, w denote the components of velocity at (x, y, z).

(ii) Show that $\phi = (x-t)(y-t)$ represents the velocity potential of an incompressible two dimensional fluid.

Show that the streamlines at time 't' are the curves $(x-t)^2 + (y-t)^2 = \text{constant}$. 5+5

Unit - 4

5. Answer ***any two*** questions :

(a) Derive the general energy equation for a control volume system. 5

(b) (i) The velocity components in a two-dimensional flow are $u = 8x^2y - \frac{8}{3}y^3$, $v = -8xy^3 + \frac{8}{3}x^3$.

Show that these velocity components represent a possible case of an incompressible flow.

(ii) Derive momentum equation from Reynold's transport theorem for fixed control volume.

2+3

(c) The velocity distribution for flow in a long circular tube of radius R is given by the one-dimensional expression

$$\vec{V} = u\hat{i} = u_{\max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i}.$$

For this profile, obtain the expression for the volume flow rate through a section normal to the axis of the tube. 5

(d) A sphere of radius a is surrounded by infinite liquid of density ρ, the pressure at infinity being Π. The sphere is suddenly annihilated. Show that the pressure at a distance r from the centre immediately falls to $\Pi\left(1 - \frac{a}{r}\right)$. 5