2024

MATHEMATICS — HONOURS

Paper : CC-13

(Metric Space and Complex Analysis)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$ denote the set of all natural, real, complex and rational numbers respectively.

Notations and symbols have their usual meanings.

1. Answer all the following multiple choice questions. For each question **1 mark** for choosing correct option and **1 mark** for justification : 2×10

(a) Which of the following d is not a metric on $\mathbb{R}$?

(i) $d(x,y)=\begin{cases}0, & \text{for } x=y\\ 1, & \text{for } x\neq y\end{cases}$ (ii) $d(x,y)=|x-y|$

(iii) $d(x,y)=\dfrac{|x-y|}{1+|x-y|}$ (iv) $d(x,y)=\big||x|-|y|\big|$.

(b) Let $Y=(0,1)\cup\{1,2\}\cup[2.5,4]$. Consider Y as metric subspace of $\mathbb{R}$ with usual metric. Which of the following is an open unit ball centred at 2?

(i) $\{2\}$ (ii) $\{1,2,3\}$

(iii) $\{2\}\cup(2.5,3)$ (iv) $\{2\}\cup[2.5,3)$.

(c) Let (X,d_1) and (Y,d_2) be metric spaces and $f:X\to Y$ be a surjective mapping. Which one of the following is true?

(i) If f is continuous, then $\{f(x_n)\}$ is Cauchy in Y whenever $\{x_n\}$ is Cauchy in X.

(ii) If f is continuous and (X,d_1) is compact, then $\{f(x_n)\}$ is Cauchy in Y whenever $\{x_n\}$ is Cauchy in X.

(iii) If $\{x_n\}$ Cauchy in X implies $\{f(x_n)\}$ is Cauchy in Y, then f is continuous.

(iv) If f is continuous and (X,d_1) is complete, then (Y,d_2) is complete.

(d) Let (X, d) be a metric space and A be a non-empty proper subset of X which is both open and closed. Then

(i) X is connected (ii) X is disconnected

(iii) X is compact (iv) X is complete.

(e) On $\mathbb{R}^2$ with usual metric, let $A = \{(x, y) : y = |x|\}$, $B = \{(x, y) : x^2 + y + 2 = 0\}$.

Then $d(A, B)$ is equal to

(i) 0 (ii) 2

(iii) 4 (iv) 1.

(f) Let $T(z) = \frac{z}{2-z}$. Then the number of fixed points of T is

(i) 0 (ii) 1

(iii) 2 (iv) infinite.

(g) Let $f: \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \operatorname{Re} Z$. Then

(i) f is differentiable at $z = 0$ only

(ii) f is everywhere differentiable

(iii) f is continuous everywhere but differentiable nowhere

(iv) f is not continuous at $z = 0$.

(h) The polar form of Cauchy-Riemann equation is :

(i) $\frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial u}{\partial \theta} = -\frac{r \partial v}{\partial r}$ (ii) $\frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial u}{\partial \theta} = r\frac{\partial v}{\partial r}$

(iii) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$ (iv) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial u}{\partial \theta} = r\frac{\partial v}{\partial r}$.

(i) $\int_{|z|=5} \frac{2\sin z}{2z+\pi} dz$ is equal to

(i) $2\pi i$ (ii) $-2\pi i$

(iii) $-\pi i$ (iv) πi.

en and

(j) The radius of convergence of the power series $\sum_{n=0}^{\infty}\left(\frac{n\sqrt{2}+i}{1+2in}\right)z^n$ is

(i) 1 (ii) ½

(iii) 0 (iv) 2.

Unit-1

(Metric Space)

Answer *any five* questions.

2. Prove or disprove :

(a) Let $d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ defined by $d(m,n) = \left|\frac{1}{m} - \frac{1}{n}\right|$ for $m, n \in \mathbb{N}$. Then d is a metric on $\mathbb{N}$.

(b) Let $R[0, 1]$ denote the class of all Riemann integrable functions from $[0, 1]$ to $\mathbb{R}$. Show that the mapping $d : R[0, 1] \times R[0, 1] \to \mathbb{R}$ defined by

$d(f,g) = \int_0^1 |f(x) - g(x)|\,dx$ for $f, g \in R[0, 1]$, is a metric on $R[0, 1]$. 3+2

3. (a) Prove that the closed ball $B[x_0, r] = \{x \in X : d(x, x_0) \le r\}$ is a closed set.

(b) Let (X, d) be a metric space and $A \subseteq X$. Show that $diam(A) = diam(\bar{A})$. 2+3

4. Let $\{x_n\}$ and $\{y_n\}$ be sequences in a metric space (X, d) and $a_n = d(x_n, y_n)$ for all $n \in \mathbb{N}$. If $\{x_n\}$ is a Cauchy sequence and $\lim_{n\to\infty} a_n = 0$, then prove that :

(a) $\{y_n\}$ is a Cauchy sequence.

(b) $\{x_n\}$ converges in X if and only if $\{y_n\}$ converges in X. 2+3

5. (a) Let (X, d_1) and (Y, d_2) be metric spaces and $f : X \to Y$, $g : X \to Y$ be continuous maps. If $F = \{x \in X : f(x) = g(x)\}$ is dense in X, then prove that $f = g$.

(b) Prove that the discrete metric space (X, d) is compact if and only if X is finite. 3+2

6. Let (X, d) be a metric space. Then prove that X is compact if and only if every collection of closed subsets of X having finite intersection property has non-empty intersection. 5

7. Let (X, d_X) be a metric space. If every continuous function $f : (X, d_X) \to (\mathbb{R}, d_u)$ has the intermediate value property (*i.e.*, if $y_1, y_2 \in f(X)$ and y is a real number between y_1 and y_2, then there exists $x \in X$ such that $f(x) = y$), then prove that (X, d_X) is a connected metric space. 5

8. (a) Let (X, d) be a metric space and $A, B \subseteq X$ are closed. If $A \cup B$ and $A \cap B$ are connected, prove that A is connected.

(b) Let (X, d) and (Y, ρ) be two metric spaces and $f: X \to Y$ be uniformly continuous on X. Let $A, B \subset X$ such that $d(A, B) = 0$. Prove that $\rho(f(A), f(B)) = 0$. 2+3

9. (a) Check whether the mapping $f: [0, 1] \to [0, 1]$ given by $f(x) = x - \frac{x^2}{2}, \forall x \in [0, 1]$ is a contraction or not, where $[0, 1]$ is equipped with usual metric of real numbers.

(b) Let (X, d) be a metric space and $x_0 \in X$. Define $f_{x_0}: X \to \mathbb{R}$ by $f_{x_0}(x) = d(x, x_0)$. Prove that f_{x_0} is a continuous. 3+2

Unit - 2

(Complex Analysis)

Answer ***any four*** questions.

10. Define stereographic projection. Prove that the stereographic projection of a circle is again a circle. 2+3

11. (a) Find $\lim\limits_{z \to 3+i} \frac{z^2}{|z|}$.

(b) Check whether the function $f: \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} |z|^2 \operatorname{Im}\left(\frac{1}{z}\right), & \text{when } z \neq 0 \\ 0, & \text{if } z = 0 \end{cases}$$

is continuous at $z = 0$ or not. 2+3

12. (a) Prove that $f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0; z = x + iy \end{cases}$

satisfies Cauchy-Riemann equation at origin but $f'(0)$ does not exist.

(b) Let $f: G \to \mathbb{C}$ be analytic, where G is a region in complex plane. If $\operatorname{Re}(f)$ is constant on G, then show that f is constant in G. 3+2

13. Let $f(z)=\sum a_n z^n$ be a power series with radius of convergence R. Show that $f(z)$ is analytic in $B(0,R)$, where $B(0,R)$ is an open ball with centre at origin of radius R. Also show that $f'(z)=\sum_{n=1}^{\infty} n a_n z^{n-1}$. 3+2

14. (a) Find the bilinear transformation which transforms $z_1=0$, $z_2=1$, $z_3=\infty$ onto $w_1=-1$, $w_2=-i$, $w_3=1$ respectively.

(b) If the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n z^n$ is R (where $0<R<\infty$), then find the radius of convergence of the power series $\sum_{n=1}^{\infty} 2^n a_n z^n$. 3+2

15. (a) State Cauchy-Goursat theorem.

(b) Find the value of the following integrals :

(i) $\displaystyle\int_{|z|=1} \frac{\sin z}{z(z-2)}dz$ (ii) $\displaystyle\int_{|z|=2} \frac{e^z}{(z-1)(z+1)}dz$ 1+(2+2)

16. (a) Let $f:\mathbb{C}\to\mathbb{C}$ be continuous on a contour γ such that $|f(z)|\le M, \forall z\in\gamma$ and let l be the length of the contour γ, then show that

$$\left|\int_{\gamma} f(z)dz\right| \le Ml.$$

(b) Find the value of the integral $\displaystyle\int_0^{1+i}\left(x-y+ix^2\right)dz$ along the straight line from $z=0$ to $z=1+i$.

3+2