2023

MATHEMATICS — HONOURS

Paper : DSE-B-2.1

(Point Set Topology)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Throughout the question, $\mathbb{R}$ and $\mathbb{N}$ denote respectively the set of real numbers and the set of natural numbers.

1. Answer all multiple choice questions. For each question, **1** mark for choosing correct option and **1** mark for justification. 2×10

(a) Let (X, τ) be a cofinite topological space, where X is an uncountable set. Then which of the following is false?

(i) Each point of X is the intersection of all of its neighbourhoods in X.

(ii) No two open sets in X are disjoint.

(iii) $\tau \subseteq \tau'$, where τ' denotes the co-countable topology on X.

(iv) There exists a metric on X which generates the topology τ on X.

(b) Let (X, τ) be a topological space and A be a proper non-empty subset of X such that $\operatorname{int}(X-A) = \phi$, (where int B denotes the interior of any subset B in X). Then which of the following is false?

(i) A is dense in X.

(ii) Every non-empty open set in X intersects A.

(iii) The only closed set in X containing A is X.

(iv) The derived set of A is an empty set.

(c) Let $\mathbb{R}$ be the set of all real numbers with usual topology and $K = \left\{\frac{1}{n} : n = 1, 2, ...\right\}$. Then K is

(i) open in $\mathbb{R}$.
(ii) closed in $\mathbb{R}$.
(iii) both open and closed in $\mathbb{R}$.
(iv) neither open nor closed in $\mathbb{R}$.

(d) The closure of the set $A = \left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, ...\right\}$ with respect to the usual topology on the set of real numbers $\mathbb{R}$ is given by

(i) $A \cup \{1\}$.
(ii) $A \cup \{2\}$.
(iii) $A \cup \{\varphi\}$.
(iv) $A \cup \{3\}$.

(e) Let $f : (\mathbb{R}, \tau_l) \to (\mathbb{R}, \tau_u)$ be defined as $f(x) = x$, $\forall x \in \mathbb{R}$, where τ_l, τ_u are the lower limit topology and the usual topology on $\mathbb{R}$ respectively, then

(i) f is not a continuous map.

(ii) f is an open map.

(iii) f is neither continuous nor an open map.

(iv) f is continuous but not an open map.

(f) Let (X, τ) be a co-countable space, where X is an uncountable set. Then which of the following is true?

(i) (X, τ) is a first countable space.

(ii) (X, τ) is a Hausdorff space.

(iii) There exists a convergent sequence in X whose limit is not unique.

(iv) A sequence $\{x_n\}$ in X is convergent if and only if there is some positive integer m such that for all $n \geqslant m$, x_n = constant.

(g) Let $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ be a topology on $X = \{a, b, c\}$. Then X is

(i) compact and Hausdorff. (ii) compact but not Hausdorff.

(iii) only Hausdorff. (iv) T_1.

(h) A continuous function f from an infinite connected space (X, τ) to a discrete two point space $\{0, 1\}$

(i) must be constant. (ii) must be non-constant.

(iii) is not closed. (iv) is not open.

(i) Let (X, τ) be an uncountable compact space and $(\mathbb{R}, \tau_u)$ be the space of real numbers with the usual topology. Then which of the following is false?

(i) There exists a continuous map $f : X \to \mathbb{R}$ which is unbounded.

(ii) A map $f : X \to \mathbb{R}$ is continuous $\Rightarrow f : X \to \mathbb{R}$ is a closed map.

(iii) If $f : X \to \mathbb{R}$ is a continuous map then $f(X)$ is closed in $\mathbb{R}$.

(iv) A map $f : X \to \mathbb{R}$ is continuous and $A \in \tau$ implies $f(X \setminus A)$ is compact in $\mathbb{R}$.

(j) Let $X = [0, 1) \cup [2, 3]$ be the subspace of the topological space $\mathbb{R}$ with the usual topology and

$f : X \to \mathbb{R}$ be a map given by $f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 2 & \text{if } 2 \le x \le 3. \end{cases}$

Then which of the following is true?

(i) f is open and continuous.

(ii) f is open but not continuous.

(iii) f is not open but continuous.

(iv) f is neither open nor continuous.

Unit - 1

(Marks : 20)

Answer ***any four*** questions.

2. (a) Let $\mathbb{N}$ be the set of natural numbers and $A_n = \{1, 2, 3, ..., n\}, n \in \mathbb{N}$. Then prove that the collector $\tau = \{A_n : n \in \mathbb{N}\} \cup \{\mathbb{N}, \varphi\}$ is a topology on $\mathbb{N}$.

   (b) Find the derived set of $\{1\}$ in the above topological space. 3+2

3. (a) Prove that the lower limit topology τ_l and the upper limit topology τ_r are both strictly finer than the usual topology τ_u on the set of all real numbers $\mathbb{R}$.

   (b) Prove that the lower limit topology τ_l and the upper limit topology τ_r on $\mathbb{R}$ are non-comparable but their intersection is the usual topology τ_u on $\mathbb{R}$. 2+3

4. Define topologically equivalent metrics on a non-empty set X. Prove that the space $(X, \tau(d))$, where $\tau(d)$ is the topology induced by a metric d on a non-empty set X is homeomorphic to the space $(X, \tau(d_1))$, where $\tau(d_1)$ is the topology induced by the metric d_1 on X, where d_1 is given by $d_1(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$, for all $x, y \in X$. 1+4

5. (a) Suppose (X, τ) is a topological space and (Y, τ_Y) is the subspace of (X, τ). Prove that, for a subset A of Y, $\overline{A}^Y = \overline{A} \cap Y$, where $\overline{A}^Y$ denotes the closure of A in (Y, τ_Y).

   (b) Find the boundary and interior of the set $\{(x, y) : x \in \mathbb{Q}\}$ in $\mathbb{R}^2$ endowed with the usual product topology. 3+2

6. Prove that in a topological space (X, τ)

   (a) the set $A \cup A'$ is the smallest closed subset containing A, where $A \subseteq X$ and A' is the derived set of A.

   (b) Prove or disprove : $\overline{A \cap B} = \overline{A} \cap \overline{B}$, where $\overline{A}$ denotes the closure of A in (X, τ). 3+2

7. (a) Let X be a non-empty set and $B = \{\{x\} : x \in X\}$. Then prove that B is a basis for a topology on X.

   (b) Give an example of a map from a topological space (X, τ) to another topological space (Y, τ') which is both open and closed but not continuous. 3+2

8. Let (X, τ) be the topological product of a family of topological spaces $\{(X_i, \tau_i) : i = 1, 2, ..., n\}$ and $p_i : X \rightarrow X_i$ denote the i-th projection map $\forall\ i = 1, 2, ..., n$. Then prove that

   (a) p_i is an open map $\forall\ i = 1, 2, ..., n$.

   (b) the product topology τ is the smallest topology on X such that each projection map is continuous. 2+3

Unit - 2

(Marks : 10)

Answer ***any two*** questions.

9. (a) Give example of a topological space which is T_1 but not T_2. Justify your answer.

(b) Prove that a topological space (X, τ) is T_1 if and only if every neighbourhood of any limit point p of any set $A \subseteq X$ intersects A in countably infinite number of points. 2+3

10. Let X be an uncountable set and p be a fixed point X. Define $\tau = \{G \subseteq X$: either $p \notin G$ or if $p \in G$ then $X \setminus G$ is finite$\}$. Prove that (X, τ) is a topological space which is not first countable. 2+3

11. (a) Let (X, τ) be a topological space and $\mathcal{B}$ a local base at $c \in X$. Prove that a sequence $\{x_n\}_n$ converges to $c \in X$ if and only if for every $B \in \mathcal{B}$, there exists a positive integer m such that for all $n \geqslant m$, $x_n \in B$.

(b) Let $f : (X, \tau_1) \to (Y, \tau_2)$ be an open, continuous surjective map, where X is first countable. Prove that Y is first countable. 3+2

12. (a) If (X_1, τ_1) and (X_2, τ_2) are two T_2 spaces, then prove that their product space (X, τ) is also a T_2 space.

(b) Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a continuous map and Y be T_2. Prove that the set $\{(x, f(x)) : x \in X\}$ is a closed set in $X \times Y$, where $X \times Y$ is endowed with the product topology. 3+2

Unit - 3

(Marks : 15)

Answer ***any three*** questions.

13. (a) Suppose (X, τ) is a topological space and $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ (Λ is an index set) is any family of closed subsets of X with the property that $\bigcap_{i=1}^{n} F_{\alpha_i} \neq \phi$ for any finite subfamily $\{F_{\alpha_i} : \alpha_i \in \Lambda, i = 1, 2, ..., n\}$ of $\mathcal{F}$. Prove that X is compact if and only if $\bigcap_{\alpha \in \Lambda} F_\alpha \neq \phi$.

(b) Let (X, τ_1) be a T_2 space and (X, τ_2) be compact such that $\tau_1 \subseteq \tau_2$. Prove that $\tau_1 = \tau_2$. 3+2

14. (a) Prove that the set of real numbers $\mathbb{R}$ with lower limit topology is disconnected.

(b) Prove that a topological space containing a dense connected set is connected. 2+3

15. (a) Prove that a real valued continuous function f on a compact space (X, τ) attains its greatest value.

(b) If K is a compact subset of a T_2 space X, then prove that K is a closed set in X. 2+3

16. (a) Prove that every closed and bounded interval of the real line $\mathbb{R}$ (i.e., $\mathbb{R}$ with the usual topology) is compact.

(b) Prove that each component of a topological space is closed.

3+2

17. (a) If every continuous real valued function on a topological space (X, τ) takes on all values between any two values that it assumes then prove that (X, τ) is connected.

(b) If A is a connected subset consisting of at least two points in a metric space (X, d) then prove that A is uncountable.

2+3

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