

Unification of some multivariate process capability indices for asymmetric specification region

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In manufacturing industries, it is often seen that the bilateral specification limits corresponding to a particular quality characteristic are not symmetric with respect to the stipulated target. A unified superstructure $C_p''(u, v)$ of univariate process capability indices was specially designed for processes with asymmetric specification limits. However, as in most of the practical situations a process consists of a number of inter-related quality characteristics, subsequently, a multivariate analogue of $C_p''(u, v)$, which is called $C_M(u, v)$, was developed. In the present paper, we study some properties of $C_M(u, v)$ like threshold value and compatibility with the asymmetry in loss function. We also discuss estimation procedures for plug-in estimators of some of the member indices of $C_M(u, v)$. Finally, the superstructure is applied to a numerical example to supplement the theory developed in this article.

Keywords and Phrases: asymmetry in loss function, capability indices, multivariate delta method, non-normality, threshold value

1 Introduction

Development of quality management practices has gradually strengthened the idea that anything produced must have come through a process where some input is converted into some useful output for which there are customers. In measuring whether such a process is capable to meet the specifications (often fixed by the customer or the designer) or not, a very popularly known measure, called *process capability index* (PCI), is used. KANE (1986), in his pioneering paper on PCIs, has established the importance of studying various univariate PCIs. Under the assumption of normality of the distribution of the quality characteristic under consideration, the four classical PCIs for symmetric specification limits are $C_p = \frac{USL - LSL}{6\sigma}$, $C_{pk} = \frac{d - |\mu - M|}{3\sigma}$, $C_{pm} = \frac{d}{3\sqrt{\sigma^2 + (\mu - T)^2}}$,

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and $C_{pmk} = \frac{d - |\mu - M|}{3\sqrt{\sigma^2 + (\mu - T)^2}}$, where USL and LSL are the upper and lower specification limits of the concerned quality characteristic, respectively, $d = \frac{USL - LSL}{2}$, $M = \frac{USL + LSL}{2}$, and T is the target.

PEARN, KOTZ and JOHNSON (1992), among others, studied the distributional properties of the existing univariate PCIs. VÄNNMAN (1995) proposed generalized univariate PCIs by developing a superstructure of univariate PCIs called $C_p(u, v)$, which is given by

$$C_p(u, v) = \frac{d - u|\mu - M|}{3\sqrt{\sigma^2 + v(\mu - T)^2}}, \quad u, v \geq 0. \quad (1)$$

However, most of the indices available in literature make an inherent assumption that the quality characteristic under consideration is symmetric about the target – which may not always be the case. The specification of a process is called asymmetric when the target value differs from the mid-point of the specification limits.

In practice, customers are often willing to allow more deviation from target towards one side of the specification interval than the other. Even if a process starts with a symmetric specification interval, frequently, it is observed that the manufacturer and the customer have to opt for an asymmetric specification interval in order to avoid unnecessary increase in the production cost. Even if data on a process parameter, whose underlying distribution is non-normal, are converted into a normal one using standard transformations, then the initial symmetric specification limits become automatically converted into asymmetric ones by virtue of the same transformation (BOYLES, 1994).

Although the presence of asymmetry in specification limits is not at all a rare event especially in manufacturing industries, the research work in this field is surprisingly fewer as compared with the case of symmetric specification limits. KANE (1986), KUSHLER and HURLEY (1992), and FRANKLIN and WASSERMAN (1992) addressed this problem by shifting either or both of the original specification limits so as to make them symmetric. However, these new limits are different from the original ones and hence often yield misleading outcomes.

BOYLES (1994) observed that most of the existing indices attain their maximum values at some point between T and M , whereas for an ideal process, the indices should be optimum at the target. To overcome the deficiencies of the indices studied earlier, he proposed a new index as $S_{pk} = S(\frac{USL - \mu}{\sigma}, \frac{\mu - LSL}{\sigma})$, where $S(x, y)$ is a smooth function, which is defined as $S(x, y) = \frac{1}{3}\Phi^{-1}\left(\frac{\Phi(x) + \Phi(y)}{2}\right)$.

VÄNNMAN (1997) proposed two superstructures of PCIs, namely, $C_{pv}(u, v)$ and $C_{pa}(u, v)$, for asymmetric specification limits. While $C_{pv}(u, v)$ fails to capture the asymmetry of the loss function, $C_{pa}(u, v)$ is not optimum on target.

PEARN (1998) proposed a new PCI analogous to C_{pk} for asymmetric tolerances, which is given by $C''_{pk} = \frac{d^* - F^*}{3\sigma}$, where $D_U = USL - T$, $D_L = T - LSL$, $d^* = \min(D_L, D_U)$, and $F^* = \max\{\frac{d^*(\mu - T)}{D_U}, \frac{d^*(T - \mu)}{D_L}\}$. PEARN *et al.* (2001) studied some properties of C^*_{pk} and proposed an estimator that is consistent and asymptotically unbiased and converges to

a mixture of two normal distributions. Later CHEN and PEARN (2001) generalized this index to a superstructure, which is defined as

$$C_p''(u, v) = \frac{d^* - uF^*}{3\sqrt{\sigma^2 + vF^2}}, \quad u \geq 0, v \geq 0, \quad (2)$$

where $F = \max\{\frac{d(\mu-T)}{D_U}, \frac{d(T-\mu)}{D_L}\}$. They compared the new index with the existing ones with respect to process yield and process center and observed the following:

1. $C_p''(u, v)$ attains maximum value when the process is on target.
2. High values of $C_p''(u, v)$ indicate high process yield, whereas small values indicate more room for improvement.

However, TAAM *et al.* (1993), among others, have pointed out that in many practical situations, a manufactured product has more than one quality characteristics to describe its features, geometric shape, and design intent. The prevailing practice of most of the industries is to calculate suitable univariate PCI for each of the quality characteristics and then suitably summarize them to have a single valued capability measure of the process. This ignores the interdependence; that is, the correlation among various quality characteristics and as such may, often, be misleading. This necessitates the use of the so-called multivariate PCIs (MPCIs). The situation becomes a little more difficult for asymmetric specification regions.

GIRI (2004) has developed a set of MPCIs for asymmetric region analogous to C_p'' , C_{pk}'' , C_{pm}'' , and C_{pmk}'' from a geometrical perspective. Suppose there are p quality characteristics, which are correlated among themselves and are having the mean vector μ and dispersion matrix Σ . Then, GIRI (2004) defined the following MPCIs:

$$\begin{aligned} \underline{C}_p'' &= \frac{3}{\sqrt{\chi_{p,0.9973}^2}} \times \inf_i (C_{p,i}''), \\ \underline{C}_{pk}'' &= \left(1 - \sup_i \left(\sup \left(\frac{\mu_i - T_i}{U_i - T_i}, \frac{T_i - \mu_i}{T_i - L_i} \right) \right)\right) \times \underline{C}_p'', \\ \underline{C}_{pm}'' &= \{1 + Q(\mu)' \Sigma^{-1} Q(\mu)\}^{-\frac{1}{2p}} \times \underline{C}_p'', \\ \underline{C}_{pmk}'' &= \{1 + (\mu - T)' \Sigma^{-1} (\mu - T)\}^{-\frac{1}{2p}} \times \underline{C}_{pk}'', \end{aligned}$$

where $C_{p,i}''$ is the C_p'' value corresponding to the i th variable, for $i = 1(1)p$; and $Q(\mu)$, the multivariate analogue of F , is a p component vector whose i th element is defined as

$$Q(\mu)_i = (\mu_i - T_i) \times \left(\frac{d}{U_i - T_i} 1_{[\mu_i > T_i]} + \frac{d}{T_i - L_i} 1_{[\mu_i < T_i]} \right), \quad i = 1(1)p,$$

$$\text{with } 1_{[\mu \in A]} = \begin{cases} 1, & \text{if } \mu \in A \\ 0, & \text{otherwise} \end{cases}.$$

Under the assumption of multivariate normality of the underlying distribution of the quality characteristics, CHATTERJEE and CHAKRABORTY (2011) have defined a

superstructure of MPCIs for asymmetric specifications, analogous to $C_p''(u, v)$ (refer to Equation 2), as

$$C_M(u, v) = \frac{1}{3} \sqrt{\frac{(\mathbf{d}^* - u\mathbf{G}^*)'\Sigma^{-1}(\mathbf{d}^* - u\mathbf{G}^*)}{1 + v\mathbf{G}'\Sigma^{-1}\mathbf{G}}}, \quad u \geq 0, v \geq 0, \quad (3)$$

where $\mathbf{d}^* = (\min(D_{1L}, D_{1U}), \min(D_{2L}, D_{2U}), \dots, \min(D_{pL}, D_{pU}))'$, that is, $d_i^* = \min(D_{iL}, D_{iU})$, for $i = 1(1)p$ with $\mathbf{D}_U = (D_{1U}, D_{2U}, \dots, D_{pU})'$ and $\mathbf{D}_L = (D_{1L}, D_{2L}, \dots, D_{pL})'$. Also $\mathbf{d} = (\frac{USL_1 - LSL_1}{2}, \frac{USL_2 - LSL_2}{2}, \dots, \frac{USL_p - LSL_p}{2})'$, that is, $d_i = \frac{USL_i - LSL_i}{2}$, for $i = 1(1)p$.

$\mathbf{G} = (a_1 d_1, a_2 d_2, \dots, a_p d_p)'$, where $a_i = [\max\{\frac{\mu_i - T_i}{D_{iU}}, \frac{T_i - \mu_i}{D_{iL}}\}]$, $\forall i = 1(1)p$. Thus, $\mathbf{G} = [\text{diag}(a_1, a_2, \dots, a_p)] \times \mathbf{d} = \mathbf{A}\mathbf{d}$, say and its univariate counterpart is given as F in formula 2.

Similarly, F^* can be generalized as $\mathbf{G}^* = \mathbf{A}\mathbf{d}^*$ for the multivariate case. Note that for $p = 1$, $C_M(u, v)$ boils down to $C_p''(u, v)$. Here boldfaced letters have been used to denote vectors.

Similar to C_p of the symmetric univariate case, $C_M(0, 0)$ also does not incorporate the process mean vector $\boldsymbol{\mu}$ in its definition, and hence it measures the true capability of a process only when it is centered on its target and fails to detect any deviation of process centering from target. Therefore, $C_M(0, 0)$ should be treated as the potential capability index rather than an actual index.

In the present article, we have made a comprehensive study of $C_M(u, v)$. In section 2, some very crucial properties of $C_M(u, v)$ are explored namely, its ability to detect asymmetry in loss function and threshold value computation. Plug-in estimators of the member indices of $C_M(u, v)$ for $u = 0, 1$ and $v = 0, 1$ along with their expectations are discussed in section 3 followed by the expressions for the corresponding expectations under a large sample scenario in section 4. Section 5 contains a numerical example showing the efficiency of $C_M(u, v)$ in dealing with asymmetric specification region in multivariate scenario. Finally, the article concludes in section 6 with a brief summary of the properties of $C_M(u, v)$ studied here.

2 Some properties of $C_M(u, v)$

2.1 $C_M(u, v)$ and asymmetry of loss function

Often asymmetry in specification limits arise owing to differences in the importance of deviation from either side of the target. Such asymmetry in specification limits makes the corresponding loss function asymmetric as well. This very phenomenon should be reflected in an ideal PCI, as deviation from target towards one specification limit is likely to incur more loss than deviation towards the other specification limit. Hence our next objective is to check the effectiveness of $C_M(u, v)$ in this respect. For this, we have adopted the approach, which was originally used by CHEN and PEARN (2001), based on the concept of 'equal departure ratio' from the target.

Let us consider two processes ' P ' and ' Q ' with the same **USL**, **LSL**, T , and Σ , but two different process means μ_P and μ_Q , respectively, such that $\mu_P > T$ and $\mu_Q < T$, satisfying (i) $\mu_P - T \neq T - \mu_Q$ and (ii) $\frac{\mu_{P_i} - T_i}{D_{U_i}} = \frac{T_i - \mu_{Q_i}}{D_{L_i}} = g_i$, say, $i = 1(1)p$; that is, each quality characteristic of the two processes P and Q has equal departure ratio from target, although μ_P and μ_Q themselves are not equidistant from either side of T . In fact, the condition $\mu_P - T \neq T - \mu_Q$ nullifies the possibility of symmetry of the loss functions of the two processes. Moreover, D_{U_i} and D_{L_i} , $i = 1(1)p$ (which are necessarily unequal owing to the asymmetry in specification limits) have been used as units to measure the deviation of mean of a particular quality characteristic from its target to incorporate the concept of asymmetry in specification region.

For processes P and Q , let us define $a_{P_i} = \frac{\mu_{P_i} - T_i}{D_{U_i}}$ and $a_{Q_i} = \frac{T_i - \mu_{Q_i}}{D_{L_i}}$, $i = 1(1)p$, respectively, such that A_P and A_Q are two p -component diagonal matrices whose i th diagonal elements are given by a_{P_i} and a_{Q_i} , respectively, for $i = 1(1)p$. Then A_P and A_Q are the A matrices corresponding to the processes P and Q , respectively. As we have already assumed $\frac{\mu_{P_i} - T_i}{D_{U_i}} = \frac{T_i - \mu_{Q_i}}{D_{L_i}}$, the A matrices of both the processes P and Q become identical, that is, $A_P = A_Q$.

Now, from Equation 3,

$$C_M(u, v) = \frac{1}{3} \sqrt{\frac{\mathbf{d}^{*'}(I_p - uA)\Sigma^{-1}(I_p - uA)\mathbf{d}^*}{1 + v\mathbf{d}'A\Sigma^{-1}A\mathbf{d}}}. \quad (4)$$

As $\mathbf{d}^*$ is not a function of the deviation of μ from T , from formula 4, $C_M(u, v)$ can be expressed as a function of the A matrix only, which we have already found to be equal for both the processes. Hence the value of $C_M(u, v)$ will remain the same for both the processes.

Thus, for two processes P and Q having identical values of **USL**, **LSL**, T , and Σ , but $\mu_P - T \neq T - \mu_Q$, if their departure ratios are equal, then these two processes will have the same $C_M(u, v)$ value. As the actual losses incurred by these two processes, namely, $\mu_P - T$ and $T - \mu_Q$, are not equal, but still their $C_M(u, v)$ values are identical, therefore, following similar logic given by CHEN and PEARNS (2001) (in the context of $C_p''(u, v)$), here also, we conclude that $C_M(u, v)$ takes the asymmetry of loss function into account.

2.2 Threshold value of $C_M(0, 0)$

Threshold value is an integral part of a PCI. If the PCI for a process is less than this value, the process is considered to be incapable of producing what it is supposed to produce. The higher the value of the PCI, the better is the process. Theoretically, the threshold value of a PCI in the univariate case is that value for which the difference between the specification limits is equal to the process spread. For univariate processes centered at the target and having symmetric specification limits with respect to the target, the threshold value of the PCI is always one. However, this is not the case with univariate processes having asymmetric specification limits (CHATTERJEE and CHAKRABORTY, 2014). Moreover, as $C_M(u, v)$, being a multivariate index, incorporates

the correlation structure among the quality characteristics into account, its threshold value should, ideally, be a function of such correlation structure.

Note that if a process is not even potentially capable, then it is advisable to look into the process and take appropriate corrective measures before production starts. Hence, generally, threshold values are computed for the PCIs like C_p and C_p'' , which measure the potential process capability. We shall now compute the threshold value of $C_M(0, 0)$ for the bivariate case. For multivariate case, it is difficult to obtain a closed form of the expression for the threshold value of $C_M(0, 0)$.

The position of μ with respect to the corresponding lower and upper specification limits play a major role in the context of threshold value computation for univariate processes with symmetric specification limits. The situation becomes more complex in the case of processes having asymmetric specification limits (both univariate and multivariate), as then one needs, in addition, to take care of the position of the target with respect to the corresponding specification limits (CHATTERJEE and CHAKRABORTY (2014) for the univariate case). Hence, while deriving the expression for the threshold value of $C_M(0, 0)$, we shall consider three different cases, depending upon the various possible positions of T with respect to **USL** and **LSL**.

We shall now derive the expression for the threshold value of $C_M(0, 0)$ for the bivariate case. For multivariate case, it is difficult to obtain a closed-form expression for the threshold value. However, it would be interesting for future study.

Case I: $D_U < D_L$, that is, $d^ = D_U$*

Let $d^* = \min(D_L, D_U) = D_U < D_L$. Then,

$$C_M(0, 0) = \frac{1}{3} \sqrt{D'_U \Sigma^{-1} D_U},$$

$$\text{that is, } C_M^2(0, 0) = \frac{1}{9} D'_U \Sigma^{-1} D_U = \frac{1}{9(1 - \rho_{12}^2)} \left[\frac{D_{2U}^2}{\sigma_2^2} + \frac{D_{1U}^2}{\sigma_1^2} - \frac{2\rho_{12}D_{1U}D_{2U}}{\sigma_1\sigma_2} \right]. \quad (5)$$

Now let $USL_1 = T_1 + k_{U_1}\sigma_1$ and $USL_2 = T_2 + k_{U_2}\sigma_2$ such that $\mathbf{USL} = \mathbf{T} + K_U \times \sigma$, where $K_U = \text{diag}(k_{U_1}, k_{U_2})$ and $\sigma = (\sigma_1, \sigma_2)'$.

Similarly, $LSL_1 = T_1 - k_{L_1}\sigma_1$ and $LSL_2 = T_2 - k_{L_2}\sigma_2$ such that $\mathbf{LSL} = \mathbf{T} - K_L \times \sigma$, where $K_L = \text{diag}(k_{L_1}, k_{L_2})$ with $K_U \neq K_L$ as otherwise the specification region will become symmetric. Also note that k_{U_i} , for $i = 1, 2$, is the distance of the USL of the i th quality characteristic from the respective target in terms of the corresponding sigma unit. The other elements of the matrices K_U and K_L may be similarly defined. Thus $D_{1U} = k_{U_1}\sigma_1$; $D_{2U} = k_{U_2}\sigma_2$ and hence

$$C_M^2(0, 0) = \frac{1}{9(1 - \rho_{12}^2)} \times \left[k_{U_1}^2 + k_{U_2}^2 - 2\rho_{12}k_{U_1}k_{U_2} \right].$$

Therefore, the threshold value of $C_M(0, 0)$ is

$$C_M^T(0, 0) = \frac{1}{3} \sqrt{\frac{k_{U_1}^2 + k_{U_2}^2 - 2\rho_{12}k_{U_1}k_{U_2}}{1 - \rho_{12}^2}}. \quad (6)$$

For $k_{U_1} = k_{U_2} = k_U$, say, that is, when both the USLs are at equal distances from their respective targets in terms of the corresponding sigma units, formula 6 simplifies to $C_M^T(0, 0) = \frac{k_U}{3} \sqrt{\frac{2}{1 + \rho_{12}}}$.

Case II: $\mathbf{D}_L < \mathbf{D}_U$, i.e. $\mathbf{d}^ = \mathbf{D}_L$*

Let $\mathbf{d}^* = \min(\mathbf{D}_L, \mathbf{D}_U) = \mathbf{D}_L \leq \mathbf{D}_U$. Then $D_{1L} = k_{L_1} \sigma_1$; $D_{2L} = k_{L_2} \sigma_2$ and hence the threshold value of $C_M(0, 0)$ is

$$C_M^T(0, 0) = \frac{1}{3} \sqrt{\frac{k_{L_1}^2 + k_{L_2}^2 - 2\rho_{12}k_{L_1}k_{L_2}}{1 - \rho_{12}^2}}. \quad (7)$$

Also for $K_{L_1} = k_{L_2} = k_L$, say, formula 7 simplifies to $C_M^T(0, 0) = \frac{k_L}{3} \sqrt{\frac{2}{1 + \rho_{12}}}$.

Case III: $D_{iL} > D_{iU}$ for one i , while for the other $D_{iU} > D_{iL}$, $i = 1, 2$

Without loss of generality, suppose $D_{1U} < D_{1L}$ and $D_{2U} > D_{2L}$. Here, $\mathbf{d}^* = \begin{pmatrix} D_{1U} \\ D_{2L} \end{pmatrix}$. Then the threshold value of $C_M(0, 0)$ will be

$$C_M^T(0, 0) = \frac{1}{3} \sqrt{\frac{k_{L_1}^2 + k_{U_2}^2 - 2\rho_{12}k_{L_1}k_{U_2}}{1 - \rho_{12}^2}}. \quad (8)$$

Interestingly, the expressions of $C_M^T(0, 0)$ for all the three cases, as obtained from Equations 6, 7, and 8, can be summarized as

$$C_M^T(0, 0) = \frac{1}{3} \sqrt{\frac{k_{.1}^2 + k_{.2}^2 - 2\rho_{12}k_{.1}k_{.2}}{1 - \rho_{12}^2}}, \quad (9)$$

where $k_{.1} = \min(k_{U_1}, k_{L_1})$ and $k_{.2} = \min(k_{U_2}, k_{L_2})$.

Hence for a 'just capable' process, that is, when $C_M(0, 0) = C_M^T(0, 0)$, from formula 9, the threshold value, being a function of ρ_{12} , is not unique (unlike univariate PCIs). Such relationship between $C_M^T(0, 0)$ and ρ_{12} is quite desirable, as unlike univariate measure, for a bivariate (or in general, multivariate) case, the variables are inter-related among themselves and hence influence the process performance according to their strength of correlation.

Also, for $k_{.1} = k_{.2} = k$, the general expression of the threshold value of $C_M(0, 0)$, under bivariate setup, as given in formula 9 boils down to

$$C_M^T(0, 0) = \frac{k}{3} \times \sqrt{\frac{2}{1 + \rho_{12}}}. \quad (10)$$

Note that in such case, the specification limits boil down to the symmetric ones. Now, when $-1 \leq \rho_{12} \leq 0$, for a just capable process, threshold value of $C_M(0, 0)$ will lie within the interval $[\frac{k\sqrt{2}}{3}, \infty)$, depending upon the position of T with respect to tolerance region. Also, for $0 \leq \rho_{12} \leq 1$, the threshold value of $C_M(0, 0)$ will lie within the interval $[\frac{k}{3}, \frac{k\sqrt{2}}{3}]$. Hence, considering these two cases together, for $-1 \leq \rho_{12} \leq 1$, the threshold value of $C_M(0, 0)$ varies between $[\frac{k}{3}, \infty)$.

Moreover, in many cases, at first, the target is fixed and then as per requirement, the upper and lower specification limits are established. Therefore, our **USL** and **LSL**, which are expressed in terms of the matrices K_1 and K_2 , are more realistic than the existing concept of $T = (3\text{USL} + \text{LSL})/4$ proposed by BOYLES (1994) and PEARNS *et al.* (2001).

The threshold values of $C_M(0, 0)$, corresponding to various combinations of (k_1, k_2) , and $\rho_{12} = -1(0.05)1$, are presented in Table 1. A wide range of (k_1, k_2) values have been covered in this table in a sense that a $k_i, i = 1, 2$, value less than 2 will make the specification design too conservative, while the value greater than 4 may overlook some serious problems present in the process.

A graphical representation of a part of Table 1 with $k_1 = 2.0$ and $k_2 = 2.0, 2.5, 3.0, 3.5, 4.0$ is given in Figure 1. Figures for other combinations of (k_1, k_2) can be generated accordingly. Table 1 and Figure 1 clearly show that for $\rho_{12} = \pm 1$, the threshold value becomes undefined. For any fixed value of (k_1, k_2) , the threshold value of $C_M(0, 0)$ is a decreasing function of the correlation coefficient (ρ_{12}) for ρ_{12} value up to 0.25, and from $\rho_{12} = 0.30$, it starts to increase. Also for any fixed value of ρ_{12} , the threshold value of $C_M(0, 0)$ reaches local minimum values for $k_1 = k_2$ and increases on either side of this point.

Finally, the threshold value of $C_M(0, 0)$ can be considered as the threshold value of $C_M(u, v)$ for $u \geq 0, v \geq 0$ in general. This is because for $\mu = M = T$, all the member MPCIs of this superstructure boil down to $C_M(0, 0)$ (CHATTERJEE and CHAKRABORTY, 2011) and the index value obtained at this stage is the minimum attainable value by a process to be capable. When $C_M(0, 0) \geq C_M^T(0, 0)$ but $C_M(u, v) < C_M^T(0, 0)$ for $(u, v) \neq (0, 0)$, the process is expected to be off target. On the other hand, if $C_M(u, v) \geq C_M^T(0, 0)$ for $u > 0$ and $v > 0$, the process, apart from being potentially capable, is also supposed to deliver the good. It is interesting to note that the same approach is followed in practice for univariate PCIs as well, where a process with $C_p(u, v) \geq 1$, for $u \geq 0, v \geq 0$, is considered to be capable, while '1' is merely the threshold value of $C_p = C_p(0, 0)$.

3 Plug-in estimators of the member indices of $C_M(u, v)$ and their estimation procedures

VÄNNMAN (1997) has rightly mentioned that only providing a capability index value is not enough for characterizing a process. It requires exploring its properties to understand the actual nature of the process. We now propose a plug-in estimator of $C_M(u, v)$, which can be defined as

$$\hat{C}_M(u, v) = \frac{1}{3} \sqrt{\frac{(\mathbf{d}^* - u\hat{\mathbf{G}}^*)' \hat{\Sigma}^{-1} (\mathbf{d}^* - u\hat{\mathbf{G}}^*)}{1 + v\hat{\mathbf{G}}' \Sigma^{-1} \hat{\mathbf{G}}}}, \quad (11)$$

Table 1. Threshold values of $C_M(0, 0)$ for $\rho_{12} = -1(0.1)1$, and $p = 2$

ρ_{12}	(k_1, k_2)															
	(2, 2)	(2, 2.5)	(2, 3)	(2, 3.5)	(2, 4)	(2.5, 2.5)	(2.5, 3)	(2.5, 3.5)	(2.5, 4)	(3, 3)	(3, 3.5)	(3, 4)	(3.5, 3.5)	(3.5, 4)	(4, 4)	
-1.00	3.667	4.134	4.616	5.108	5.608	4.584	5.050	5.528	6.015	5.501	5.965	6.441	6.917	7.394	7.871	
-0.95	3.667	4.134	4.616	5.108	5.608	4.584	5.050	5.528	6.015	5.501	5.965	6.441	6.917	7.394	7.871	
-0.90	2.604	2.937	3.280	3.631	3.988	3.256	3.582	3.927	4.275	3.907	4.237	4.576	4.915	5.254	5.593	
-0.85	2.136	2.409	2.692	2.981	3.276	2.671	2.942	3.223	3.509	3.205	3.476	3.754	4.039	4.324	4.609	
-0.80	1.859	2.097	2.344	2.597	2.854	2.324	2.561	2.805	3.056	2.789	3.025	3.268	3.514	3.761	4.008	
-0.75	1.671	1.886	2.108	2.337	2.570	2.089	2.302	2.523	2.749	2.507	2.720	2.938	3.157	3.377	3.597	
-0.70	1.534	1.731	1.936	2.147	2.362	1.917	2.113	2.316	2.524	2.301	2.496	2.697	2.899	3.101	3.303	
-0.65	1.428	1.612	1.803	2.000	2.202	1.785	1.968	2.157	2.352	2.142	2.324	2.512	2.699	2.886	3.073	
-0.60	1.344	1.517	1.698	1.884	2.075	1.680	1.852	2.031	2.215	2.016	2.187	2.364	2.542	2.720	2.898	
-0.55	1.275	1.439	1.611	1.789	1.972	1.593	1.757	1.927	2.102	1.912	2.075	2.244	2.421	2.598	2.775	
-0.50	1.217	1.374	1.540	1.710	1.886	1.521	1.678	1.841	2.009	1.826	1.981	2.143	2.305	2.467	2.629	
-0.45	1.168	1.320	1.479	1.644	1.813	1.461	1.611	1.768	1.930	1.753	1.902	2.058	2.216	2.374	2.532	
-0.40	1.127	1.273	1.427	1.587	1.752	1.409	1.554	1.706	1.863	1.690	1.835	1.985	2.136	2.287	2.438	
-0.35	1.091	1.233	1.383	1.539	1.699	1.364	1.504	1.652	1.806	1.636	1.776	1.923	2.069	2.216	2.363	
-0.30	1.060	1.198	1.344	1.497	1.654	1.325	1.462	1.606	1.756	1.590	1.726	1.869	1.985	2.120	2.254	
-0.25	1.033	1.167	1.311	1.461	1.615	1.291	1.425	1.566	1.713	1.549	1.682	1.822	1.940	2.066	2.192	
-0.20	1.009	1.141	1.282	1.429	1.581	1.261	1.392	1.531	1.673	1.514	1.644	1.780	1.906	2.026	2.146	
-0.15	0.989	1.118	1.257	1.402	1.552	1.236	1.364	1.501	1.643	1.483	1.611	1.745	1.877	1.997	2.117	
-0.10	0.971	1.098	1.235	1.379	1.528	1.214	1.340	1.475	1.615	1.456	1.582	1.715	1.847	1.977	2.107	
-0.05	0.956	1.081	1.217	1.360	1.507	1.195	1.319	1.452	1.592	1.433	1.557	1.689	1.824	1.942	2.069	
0.00	0.943	1.067	1.202	1.344	1.491	1.179	1.302	1.434	1.572	1.414	1.537	1.667	1.796	1.911	2.034	
0.05	0.932	1.055	1.189	1.331	1.476	1.165	1.287	1.418	1.556	1.398	1.519	1.649	1.772	1.886	2.004	
0.10	0.924	1.046	1.180	1.321	1.468	1.154	1.276	1.406	1.544	1.385	1.506	1.634	1.757	1.864	1.977	
0.15	0.917	1.039	1.173	1.314	1.462	1.144	1.267	1.398	1.536	1.376	1.495	1.624	1.746	1.847	1.957	
0.20	0.913	1.033	1.167	1.311	1.459	1.141	1.261	1.392	1.531	1.369	1.489	1.617	1.739	1.846	1.954	
0.25	0.911	1.031	1.165	1.310	1.456	1.139	1.259	1.390	1.530	1.366	1.486	1.615	1.737	1.844	1.952	
0.30	0.911	1.031	1.165	1.310	1.456	1.139	1.259	1.390	1.530	1.366	1.486	1.615	1.737	1.844	1.952	
0.35	0.914	1.034	1.168	1.312	1.457	1.143	1.260	1.391	1.533	1.367	1.487	1.616	1.739	1.846	1.954	
0.40	0.918	1.038	1.172	1.316	1.461	1.147	1.264	1.394	1.537	1.371	1.491	1.620	1.742	1.849	1.957	
0.45	0.929	1.045	1.184	1.324	1.468	1.154	1.273	1.403	1.541	1.380	1.500	1.629	1.749	1.859	1.967	
0.50	0.943	1.071	1.217	1.374	1.521	1.162	1.282	1.429	1.573	1.394	1.517	1.652	1.774	1.886	1.994	
0.55	0.961	1.093	1.243	1.405	1.576	1.178	1.305	1.447	1.599	1.414	1.540	1.678	1.816	1.949	2.081	
0.60	0.986	1.122	1.277	1.446	1.624	1.202	1.336	1.477	1.630	1.442	1.570	1.712	1.852	1.985	2.117	
0.65	1.019	1.160	1.323	1.500	1.687	1.274	1.413	1.570	1.741	1.539	1.666	1.819	1.978	2.131	2.283	
0.70	1.064	1.213	1.385	1.572	1.771	1.330	1.476	1.642	1.833	1.597	1.740	1.902	2.063	2.224	2.385	
0.75	1.127	1.285	1.469	1.671	1.886	1.409	1.563	1.741	1.915	1.690	1.843	2.016	2.180	2.344	2.508	
0.80	1.217	1.389	1.591	1.813	2.049	1.521	1.690	1.884	2.097	1.826	1.991	2.180	2.375	2.569	2.764	
0.85	1.357	1.580	1.778	2.031	2.299	1.696	1.885	2.105	2.346	2.036	2.221	2.434	2.627	2.882	3.088	
0.90	1.604	1.834	2.108	2.412	2.736	2.005	2.230	2.493	2.784	2.406	2.627	2.882	3.087	3.308	3.529	
0.95	2.188	2.503	2.884	3.308	3.759	2.735	3.043	3.407	3.812	3.382	3.585	3.937	4.128	4.376	4.624	
1.00	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	

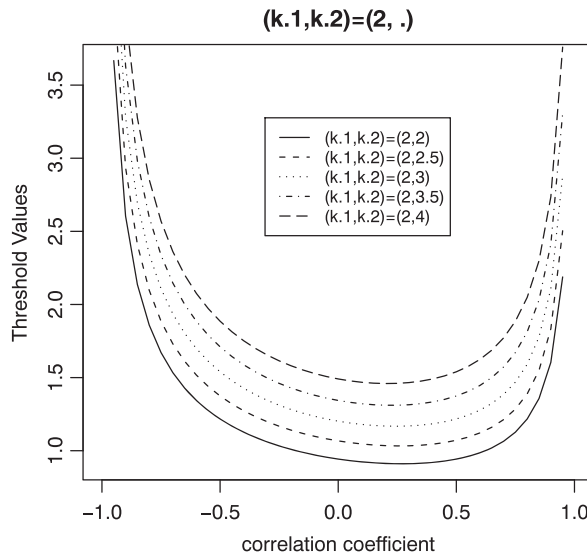


Fig. 1. Threshold values of $C_M(0, 0)$ for various values of ρ_{12} , $k_{.1} = 2.0$ and $k_{.2} = 2.0, 2.5, 3.0, 3.5, 4.0$.

where $\hat{\mathbf{G}} = \hat{A}\mathbf{d}$ and $\hat{\mathbf{G}}^* = \hat{A}\mathbf{d}^*$, where $\hat{A} = ((\hat{a}_i))$ and $\hat{a}_i = [\max\{\frac{\bar{X}_i - T_i}{D_{iU}}, \frac{T_i - \bar{X}}{D_{iL}}\}]$. For computational simplicity, here we shall consider only three combinations of the (u, v) values, namely, $(0, 0)$, $(1, 0)$, and $(0, 1)$.

3.1 Expectation of the plug-in estimator of $C_M(0, 0)$

From formula 11, a plug-in estimator of $C_M(0, 0)$ is

$$\hat{C}_M(0, 0) = \frac{1}{3} \times \sqrt{\mathbf{d}^{*'} \hat{\Sigma}^{-1} \mathbf{d}^*} = \frac{1}{n} \times \sqrt{(n-1) \mathbf{d}^{*'} S^{*-1} \mathbf{d}^*}, \quad (12)$$

where S^* is the sum of squares–sum of product matrix and $S^* = (n-1) \hat{\Sigma}^{-1} \sim W_p(n-1, \Sigma)$ with $W_p(n-1, \Sigma)$ denoting p -variate Wishart distribution with parameters $(n-1)$ and Σ . Hence, from formula 12,

$$\hat{C}_M(0, 0) \sim \{C_M(0, 0)\} \times \sqrt{n-1} \times \chi_{n-p}^{-1}, \quad (13)$$

where χ_{n-p}^{-1} denotes the inverse chi distribution with $n-p$ degrees of freedom. In this context, the inverse chi-square distribution is a continuous probability distribution of a positive valued random variable whose reciprocal (multiplicative inverse) follows chi-square distribution. The values of the density functions and the random number generators of the inverse chi-square distribution are available in *geoR* package of the R software (RIBEIRO JR and DIGGLE, 2001).

Thus, for $b_{n-p} = \frac{2}{n-1} \times \frac{\Gamma(\frac{n-p-1}{2})}{\Gamma(\frac{n-p}{2})}$,

$$E[\hat{C}_M(0, 0)] = \sqrt{n-1} \times \left\{ \frac{\Gamma(\frac{n-p-1}{2})}{\Gamma(\frac{n-p}{2})} \right\} \times C_M(0, 0), \quad (14)$$

$$\text{which implies, } E[\tilde{C}_M(0, 0)] = E[b_{n-p} \hat{C}_M(0, 0)] = C_M(0, 0). \quad (15)$$

Thus, $\tilde{C}_M(0, 0)$ is an unbiased estimator of $C_M(0, 0)$.

3.2 Expectation of the plug-in estimator of $C_M(1, 0)$

From formula 11, the plug-in estimator of $C_M(1, 0)$ is

$$\begin{aligned} \hat{C}_M(1, 0) &= \frac{1}{3} \times \sqrt{(\mathbf{d}^* - \hat{\mathbf{G}}^*)' \hat{\Sigma}^{-1} (\mathbf{d}^* - \hat{\mathbf{G}}^*)} \\ &= \frac{\sqrt{n-1}}{3} \times \sqrt{(\mathbf{d}^* - \hat{\mathbf{G}}^*)' S^{*-1} (\mathbf{d}^* - \hat{\mathbf{G}}^*)}, \end{aligned} \quad (16)$$

where $\hat{\mathbf{G}}^* = \hat{A}\mathbf{d}^*$ is a $(q \times 1)$ vector whose i th element is given by $\hat{G}_i^* = \hat{a}_i d_i^* = \max[\bar{X}_i - T_i, (T_i - \bar{X}_i) \times \frac{d_{ui}}{d_{li}}]$, for $i = 1(1)p$.

As $\hat{\mathbf{G}}^*$ is a function of $\bar{\mathbf{X}}$ and hence is a random vector, $(\mathbf{d}^* - \hat{\mathbf{G}}^*) = \mathbf{H}^*$ will also be a random vector with a certain probability distribution. Then, for some particular value of $\mathbf{H}^*$, say $\mathbf{h}^*$, we have

$$\begin{aligned} \hat{C}_M(1, 0)|_{\mathbf{H}^* = \mathbf{h}^*} &= \frac{1}{3} \sqrt{\mathbf{h}^{*'} \hat{\Sigma}^{-1} \mathbf{h}^*} \\ &= \frac{\sqrt{n-1}}{3} \sqrt{\mathbf{h}^{*'} S^{*-1} \mathbf{h}^*} \\ &= \frac{\sqrt{n-1}}{3} \sqrt{\mathbf{h}^{*'} \Sigma^{-1} \mathbf{h}^*} \times \sqrt{\frac{\mathbf{h}^{*'} S^{*-1} \mathbf{h}^*}{\mathbf{h}^{*'} \Sigma^{-1} \mathbf{h}^*}}. \end{aligned} \quad (17)$$

Now, $\frac{\mathbf{h}^{*'} S^{*-1} \mathbf{h}^*}{\mathbf{h}^{*'} \Sigma^{-1} \mathbf{h}^*} \sim \chi^2_{n-p}^{-1}$, where $\chi^2_{n-p}^{-1}$ stands for inverse chi-square distribution with $(n - q)$ degrees of freedom (Giri, 2004), and this is independent of the choice of $\mathbf{h}^*$ and hence of $\mathbf{h}^{*'} \Sigma^{-1} \mathbf{h}^*$. Thus, from formula 17,

$$\begin{aligned} E[\hat{C}_M(1, 0)] &= E_{\mathbf{H}^*} \left[E(\hat{C}_M(1, 0)|_{\mathbf{H}^* = \mathbf{h}^*}) \right] \\ &= \frac{\sqrt{n-1}}{3} \times E_{\mathbf{H}^*} \left[E \left(\sqrt{\mathbf{H}^{*'} \Sigma^{-1} \mathbf{H}^*} \times \sqrt{\frac{\mathbf{H}^{*'} S^{*-1} \mathbf{H}^*}{\mathbf{H}^{*'} \Sigma^{-1} \mathbf{H}^*}} \middle|_{\mathbf{H}^* = \mathbf{h}^*} \right) \right] \\ &= \frac{\sqrt{n-1}}{3} \times E_{\mathbf{H}^*} \left[\sqrt{\mathbf{H}^{*'} \Sigma^{-1} \mathbf{H}^*} \right] \times E \left[\sqrt{\chi^2_{n-p}^{-1}} \right]. \end{aligned} \quad (18)$$

Again,

$$E \left[\chi_{n-p}^{-1} \right] = \frac{1}{\sqrt{2}} \times \frac{\Gamma \left(\frac{n-p-1}{2} \right)}{\Gamma \left(\frac{n-p}{2} \right)}. \quad (19)$$

Hence

$$E \left[\hat{C}_M(1, 0) \right] = \frac{\sqrt{n-1}}{3\sqrt{2}} \times \frac{\Gamma \left(\frac{n-q-1}{2} \right)}{\Gamma \left(\frac{n-q}{2} \right)} \times E \left[\sqrt{\mathbf{H}^{*'} \Sigma^{-1} \mathbf{H}^*} \right]. \quad (20)$$

Note that derivation of the exact expression of $E \left[\sqrt{\mathbf{H}^{*'} \Sigma^{-1} \mathbf{H}^*} \right]$ will be complicated.

We shall now study the nature of bias associated to $E \left[\hat{C}_M(1, 0) \right]$ through a simulation study. For this, we consider a hypothetical process having two correlated quality characteristics. The upper and lower specification limits for the first quality characteristic are, say, $U_1 = 200$ and $L_1 = 100$, respectively, while for the second quality characteristic, these specification limits are $U_2 = 350$ and $L_2 = 220$, respectively. Thus, $d_1 = 50$, $M_1 = 150$, $d_2 = 65$, and $M_2 = 285$.

For this specification region, let us consider two different sets of target values. For the first case, let the target values for the two quality characteristics be $T_1 = 135$ and $T_2 = 300$, respectively. This will make the specification region moderately asymmetric with respect to the corresponding target vector. Again, for the second case, let $T_1 = 200$ and $T_2 = 230$, respectively. Thus in this case, the target vector, namely, $\mathbf{T}$, highly deviates from $\mathbf{M}$, and hence the specification region is highly asymmetric with respect to $\mathbf{T}$.

Now, for each of these two $\mathbf{T}$ values, we simulate bivariate normal data with the following mean vectors and the dispersion matrices, each for 10,000 times:

1. $\boldsymbol{\mu} = \begin{pmatrix} 140 \\ 210 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 20 & 21.16 \\ 21.16 & 35 \end{pmatrix}$. Here, $\mathbf{T} < \boldsymbol{\mu} < \mathbf{M}$ and the quality characteristics are *highly* correlated with the correlation coefficient $\rho = 0.8$.

2. $\boldsymbol{\mu} = \begin{pmatrix} 140 \\ 210 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 20 & 13.23 \\ 13.23 & 35 \end{pmatrix}$.

Here also $\mathbf{T} < \boldsymbol{\mu} < \mathbf{M}$ and the quality characteristics are *moderately* correlated with the correlation coefficient $\rho = 0.5$.

3. $\boldsymbol{\mu} = \begin{pmatrix} 115 \\ 250 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 20 & 21.16 \\ 21.16 & 35 \end{pmatrix}$.

Here, $\boldsymbol{\mu} < \mathbf{T} < \mathbf{M}$ and the quality characteristics are *highly* correlated with the correlation coefficient $\rho = 0.8$.

4. $\boldsymbol{\mu} = \begin{pmatrix} 115 \\ 250 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 20 & 13.23 \\ 13.23 & 35 \end{pmatrix}$.

Here also, $\boldsymbol{\mu} < \mathbf{T} < \mathbf{M}$ and the quality characteristics are *moderately* correlated with the correlation coefficient $\rho = 0.5$.

5. $\mu = \begin{pmatrix} 170 \\ 330 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 20 & 21.16 \\ 21.16 & 35 \end{pmatrix}$.

Here, $T < M < \mu$ and the quality characteristics are *highly* correlated with the correlation coefficient $\rho = 0.8$.

6. $\mu = \begin{pmatrix} 170 \\ 330 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 20 & 13.23 \\ 13.23 & 35 \end{pmatrix}$.

Here also, $T < M < \mu$ and the quality characteristics are *moderately* correlated with the correlation coefficient $\rho = 0.5$.

Thus we have considered three different positions of μ with respect to M and T .
Now, for moderately asymmetric specification region and for the aforementioned six cases, let the bias incurred owing to estimation of $C_M(1, 0)$ by $\hat{C}_M(1, 0)$ be denoted by $\text{bias}^{(1)}$, $\text{bias}^{(2)}$, $\text{bias}^{(3)}$, $\text{bias}^{(4)}$, $\text{bias}^{(5)}$, and $\text{bias}^{(6)}$. Table 2 depicts the relationship between the sample size (n) and the bias incurred owing to estimating $C_M(1, 0)$ by $\hat{C}_M(1, 0)$ for this case.

Again, for highly asymmetric specification region and for the aforementioned six cases, let the bias incurred owing to estimation of $C_M(1, 0)$ by $\hat{C}_M(1, 0)$ be denoted by $\text{bias}^{(7)}$, $\text{bias}^{(8)}$, $\text{bias}^{(9)}$, $\text{bias}^{(10)}$, $\text{bias}^{(11)}$, and $\text{bias}^{(12)}$. Table 3 depicts the relationship between the sample size (n) and the bias incurred by estimating $C_M(1, 0)$ by $\hat{C}_M(1, 0)$ for this case.

Similar tables can be constructed for other values of U , L , T , μ , and Σ as well.
From Tables 2 and 3, the following can be observed:

- 1. The bias incurred owing to estimation of $C_M(1, 0)$ by $\hat{C}_M(1, 0)$ is, in general, negative.

Table 2. Relationship between the sample size (n) and the bias incurred owing to estimating $C_M(1, 0)$ by $\hat{C}_M(1, 0)$ for moderately asymmetric specification region

n	10	30	50	100	150	200	250	300
$\text{bias}^{(1)}$	-0.4497	-0.1182	-0.0681	-0.0331	-0.0218	-0.0163	-0.0130	-0.0108
$\text{bias}^{(2)}$	-0.4924	-0.1295	-0.0746	-0.0362	-0.0239	-0.0178	-0.0142	-0.0118
$\text{bias}^{(3)}$	-0.2012	-0.0529	-0.0305	-0.0148	-0.0098	-0.0073	-0.0058	-0.0048
$\text{bias}^{(4)}$	-0.2199	-0.0578	-0.0333	-0.0162	-0.0107	-0.0080	-0.0063	-0.0053
$\text{bias}^{(5)}$	-0.2160	-0.0568	-0.0327	-0.0152	-0.0105	-0.0078	-0.0062	-0.0052
$\text{bias}^{(6)}$	-0.2359	-0.0620	-0.0357	-0.0173	-0.0114	-0.0085	-0.0068	-0.0057

Table 3. Relationship between the sample size (n) and the bias incurred owing to estimating $C_M(1, 0)$ by $\hat{C}_M(1, 0)$ for highly asymmetric specification region

n	10	30	50	100	150	200	250	300
$\text{bias}^{(7)}$	-0.0822	-0.0216	-0.0124	-0.0060	-0.0040	-0.0030	-0.0024	-0.0020
$\text{bias}^{(8)}$	-0.0569	-0.0150	-0.0086	-0.0042	-0.0028	-0.0021	-0.0016	-0.0014
$\text{bias}^{(9)}$	-0.1370	-0.0360	-0.0207	-0.0101	-0.0066	-0.0050	-0.0040	-0.0033
$\text{bias}^{(10)}$	-0.0949	-0.0249	-0.0144	-0.0070	-0.0046	-0.0034	-0.0027	-0.0023
$\text{bias}^{(11)}$	-0.0274	-0.0072	-0.0041	-0.0020	-0.0013	-0.0010	-0.0008	-0.0006
$\text{bias}^{(12)}$	-0.0190	-0.0050	-0.0029	-0.0014	-0.0009	-0.0007	-0.0005	-0.0004

2. The bias decreases with the increase in the sample size (n).
3. For $C_M(1, 0)$, the bias decreases substantially for high values of sample sizes.
4. The performance of $\hat{C}_M(1, 0)$ is better; that is, the bias is considerably smaller, for highly asymmetric specification regions, as compared with the moderately asymmetric specification regions, irrespective of the sample size.
5. The bias decreases with the increase in the value of the correlation coefficient (ρ). This implies that $\hat{C}_M(1, 0)$ performs better for the processes, for which the quality characteristics are highly correlated. This is certainly a desirable property, as the motivation for defining $C_M(u, v)$, in general, was to assess the capability of multivariate processes having correlated quality characteristics and where the specification region is asymmetric with respect to the target vector.

3.3 Plug-in estimator of $C_M(0, 1)$ as its consistent estimator

From formula 11, the plug-in estimator of $C_M(0, 1)$ is

$$\hat{C}_M(0, 1) = \frac{1}{3} \sqrt{\frac{\mathbf{d}^{*'} \hat{\Sigma}^{-1} \mathbf{d}^*}{1 + \hat{\mathbf{G}}' \Sigma^{-1} \hat{\mathbf{G}}}}, \quad (21)$$

where $\hat{\mathbf{G}} = \hat{\mathbf{A}}\mathbf{d}$ is a $(q \times 1)$ vector whose i th element is given by

$$\hat{G}_i = \hat{a}_i d_i = \max[\bar{X}_i - T_i, (T_i - \bar{X}_i)] \times d_i, \text{ for } i = 1(1)p.$$

Now, following (CASELLA and BERGER, 2007), as $\bar{\mathbf{X}}$ and $\hat{\Sigma}$ are consistent estimators of $\boldsymbol{\mu}$ and Σ , for a continuous function, $h(\bar{\mathbf{X}}, \hat{\Sigma}) \rightarrow h(\boldsymbol{\mu}, \Sigma)$ almost surely. Thus, from formula 21,

$$\hat{C}_M(0, 1) \rightarrow \frac{1}{3} \sqrt{\frac{\mathbf{d}^{*'} \Sigma^{-1} \mathbf{d}^*}{1 + \mathbf{G}' \Sigma^{-1} \mathbf{G}}} = C_M(0, 1), \text{ almost surely.} \quad (22)$$

Thus, $\hat{C}_M(0, 1)$ is a consistent estimator of $C_M(0, 1)$.

4 Asymptotic expectations of the plug-in estimators of the member indices of

$C_M(u, v)$, for $u = 0, 1$ and $v = 0, 1$

Apart from $E[\hat{C}_M(0, 0)]$, the derivation of $E[\hat{C}_M(u, v)]$, for $u = 0, 1$ and $v = 0, 1$, is complicated, and it is difficult to obtain any closed-form expression for them. However, for sufficiently large samples, $E[\hat{C}_M(u, v)]$ can be obtained using the multivariate δ method (CASELLA and BERGER, 2007) separately for the following:

1. $\boldsymbol{\mu} > \mathbf{T}$;
2. $\boldsymbol{\mu} < \mathbf{T}$;
3. $\mu_i > T_i$, for some $i = 1(1)p$ and for others, $\mu_i < T_i$; that is, here we consider a mixture of cases I and II.

Let

$$g(\boldsymbol{\mu}, \Sigma) = C_M(u, v) = \frac{1}{3} \sqrt{\frac{M(\boldsymbol{\mu}, \Sigma)}{N(\boldsymbol{\mu}, \Sigma)}}, \quad (23)$$

where

$$M(\boldsymbol{\mu}, \Sigma) = (\mathbf{d}^* - u \mathbf{G}^*)' \Sigma^{-1} (\mathbf{d}^* - u \mathbf{G}^*), \quad (24)$$

and

$$N(\boldsymbol{\mu}, \Sigma) = 1 + v \mathbf{G}' \Sigma^{-1} \mathbf{G}. \quad (25)$$

Note that

$$\frac{\delta g(\boldsymbol{\mu}, \Sigma)}{\delta \boldsymbol{\mu}} = \frac{1}{6} \times \left\{ \frac{N(\boldsymbol{\mu}, \Sigma) \frac{\delta M(\boldsymbol{\mu}, \Sigma)}{\delta \boldsymbol{\mu}} - M(\boldsymbol{\mu}, \Sigma) \frac{\delta N(\boldsymbol{\mu}, \Sigma)}{\delta \boldsymbol{\mu}}}{[M(\boldsymbol{\mu}, \Sigma) N^3(\boldsymbol{\mu}, \Sigma)]^{\frac{1}{2}}} \right\}, \quad (26)$$

and

$$\frac{\delta g(\boldsymbol{\mu}, \Sigma)}{\delta \Sigma} = \frac{1}{6} \times \left\{ \frac{N(\boldsymbol{\mu}, \Sigma) \frac{\delta M(\boldsymbol{\mu}, \Sigma)}{\delta \Sigma} - M(\boldsymbol{\mu}, \Sigma) \frac{\delta N(\boldsymbol{\mu}, \Sigma)}{\delta \Sigma}}{[M(\boldsymbol{\mu}, \Sigma) N^3(\boldsymbol{\mu}, \Sigma)]^{\frac{1}{2}}} \right\}. \quad (27)$$

CASE 1. When $\boldsymbol{\mu} > \mathbf{T}$, that is, $\mu_i > T_i$, for $i = 1(1)q$

Here, the i th diagonal element of A is $a_i = \frac{\mu_i - T_i}{d_{ui}}$, for $i = 1(1)p$, and hence A can be redefined as $A = A_1 A_2$, where both A_1 and A_2 are $(p \times p)$ diagonal matrices with A_1 having $(\mu_i - T_i)$ as its i th diagonal element and A_2 having $\frac{1}{d_{ui}}$ as its i th diagonal element, for $i = 1(1)p$. Then in the present context, for $u = 0, 1$, the expression of $M(\boldsymbol{\mu}, \Sigma)$ becomes

$$\begin{aligned} M(\boldsymbol{\mu}, \Sigma) &= (\mathbf{d}^* - u \mathbf{G}^*)' \Sigma^{-1} (\mathbf{d}^* - u \mathbf{G}^*) \\ &= (\mathbf{d}^* - u A \mathbf{d}^*)' \Sigma^{-1} (\mathbf{d}^* - u A \mathbf{d}^*) \\ &= (\mathbf{d}^* - u A_1 A_2 \mathbf{d}^*)' \Sigma^{-1} (\mathbf{d}^* - u A_1 A_2 \mathbf{d}^*). \end{aligned} \quad (28)$$

$$\begin{aligned} \text{This implies that } \frac{\delta M(\boldsymbol{\mu}, \Sigma)}{\delta \boldsymbol{\mu}} &= -2u \mathbf{d}^{*'} A_2 \Sigma^{-1} (I_q - u A_1 A_2) \mathbf{d}^* \\ &= -2u \mathbf{d}^{*'} A_2 \Sigma^{-1} (\mathbf{d}^* - u A \mathbf{d}^*). \end{aligned} \quad (29)$$

Similarly for $v = 0, 1$,

$$N(\boldsymbol{\mu}, \Sigma) = 1 + v \mathbf{d}' A_2 A_1 \Sigma^{-1} A_1 A_2 \mathbf{d}. \quad (30)$$

$$\begin{aligned} \text{This implies that } \frac{\delta N(\boldsymbol{\mu})}{\delta \boldsymbol{\mu}} &= 2v \mathbf{d}' A_2 \Sigma^{-1} A_1 A_2 \mathbf{d} \\ &= 2v \mathbf{d}' A_2 \Sigma^{-1} A \mathbf{d}. \end{aligned} \quad (31)$$

Again, for $u = 0, 1$,

$$\frac{\delta M(\boldsymbol{\mu}, \Sigma)}{\delta \Sigma} = -(\Sigma^{-1})'(\mathbf{d}^* - u \mathbf{G}^*)(\mathbf{d}^* - u \mathbf{G}^*)'(\Sigma^{-1})', \quad (32)$$

and for $v = 0, 1$,

$$\begin{aligned} \frac{\delta N(\boldsymbol{\mu}, \Sigma)}{\delta \Sigma} &= \frac{\delta}{\delta \Sigma} [1 + v \mathbf{G}' \Sigma^{-1} \mathbf{G}] \\ &= -v (\Sigma^{-1})' \mathbf{G} \mathbf{G}' (\Sigma^{-1})'. \end{aligned} \quad (33)$$

CASE 2. When $\boldsymbol{\mu} < \mathbf{T}$, that is, $\mu_i < T_i$, for $i = 1(1)p$

Here, the i th diagonal element of ' A ' is $a_i = \frac{T_i - \mu_i}{d_i}$, for $i = 1(1)p$, and hence A can be redefined as $A = A_3 A_4$, where both A_3 and A_4 are $(p \times p)$ diagonal matrices with A_3 having $(T_i - \mu_i)$ as its i th diagonal element and A_4 having $\frac{1}{d_i}$ as its i th diagonal element, for $i = 1(1)p$.

Then for $\boldsymbol{\mu} < \mathbf{T}$ and for $u = 0, 1$, the expression of $M(\boldsymbol{\mu}, \Sigma)$ becomes

$$\begin{aligned} M(\boldsymbol{\mu}, \Sigma) &= (\mathbf{d}^* - u \mathbf{G}^*)' \Sigma^{-1} (\mathbf{d}^* - u \mathbf{G}^*) \\ &= (\mathbf{d}^* - u A \mathbf{d}^*)' \Sigma^{-1} (\mathbf{d}^* - u A \mathbf{d}^*) \\ &= (\mathbf{d}^* - u A_3 A_4 \mathbf{d}^*)' \Sigma^{-1} (\mathbf{d}^* - u A_3 A_4 \mathbf{d}^*). \end{aligned} \quad (34)$$

$$\begin{aligned} \text{This implies that } \frac{\delta M(\boldsymbol{\mu}, \Sigma)}{\delta \boldsymbol{\mu}} &= -2u \mathbf{d}^{*'} A_4 \Sigma^{-1} (I_q - u A_3 A_4) \mathbf{d}^* \\ &= -2u \mathbf{d}^{*'} A_4 \Sigma^{-1} (\mathbf{d}^* - u A \mathbf{d}^*). \end{aligned} \quad (35)$$

Similarly for $v = 0, 1$,

$$N(\boldsymbol{\mu}, \Sigma) = 1 + v \mathbf{d}' A_3 A_4 \Sigma^{-1} A_3 A_4 \mathbf{d}. \quad (36)$$

$$\begin{aligned} \text{This implies that } \frac{\delta N(\boldsymbol{\mu}, \Sigma)}{\delta \boldsymbol{\mu}} &= 2v \mathbf{d}' A_4 \Sigma^{-1} A_3 A_4 \mathbf{d} \\ &= 2v \mathbf{d}' A_4 \Sigma^{-1} A \mathbf{d}. \end{aligned} \quad (37)$$

Interestingly, because the case is based upon the relative position of $\boldsymbol{\mu}$ with respect to $\mathbf{M}$ and because Σ is independent of the change in origin, the expressions for $\frac{\delta M(\boldsymbol{\mu}, \Sigma)}{\delta \Sigma}$ and $\frac{\delta N(\boldsymbol{\mu}, \Sigma)}{\delta \Sigma}$ will remain the same as in case I.

The third case, that is, when for some ' i ' $\mu_i > T_i$, while for others $\mu_i < T_i$, for $i = 1(1)p$, the computations for $\frac{\delta M(\boldsymbol{\mu}, \Sigma)}{\delta \boldsymbol{\mu}}$ and $\frac{\delta N(\boldsymbol{\mu}, \Sigma)}{\delta \boldsymbol{\mu}}$ are laborious but can be carried out following the previous two cases. Moreover, because the expressions for $\frac{\delta M(\boldsymbol{\mu}, \Sigma)}{\delta \Sigma}$ and $\frac{\delta N(\boldsymbol{\mu}, \Sigma)}{\delta \Sigma}$ were the same for both cases I and II, those will remain the same for case III as well.

Thus using the expressions for $\frac{\delta M(\boldsymbol{\mu}, \Sigma)}{\delta \boldsymbol{\mu}}$, $\frac{\delta N(\boldsymbol{\mu}, \Sigma)}{\delta \boldsymbol{\mu}}$, $\frac{\delta M(\boldsymbol{\mu}, \Sigma)}{\delta \Sigma}$, and $\frac{\delta N(\boldsymbol{\mu}, \Sigma)}{\delta \Sigma}$, as obtained from the aforementioned three cases and from the 'multivariate δ method' (CASELLA and BERGER, 2007), we can conclude that for sufficiently large sample and for $u = 0, 1$ and $v = 0, 1$, $E \left[\hat{C}_M(u, v) \right] = g(\boldsymbol{\mu}, \Sigma) = C_M(u, v)$.

5 A real-life example

We are now in a position to discuss the properties of $C_M(u, v)$, which we have already studied, in the light of a real-life example. To demonstrate the performance of $C_M(u, v)$, for $u = 0, 1$ and $v = 0, 1$, we consider the data set that was originally used by SULTAN (1986). Several authors like CHAN, CHENG and SPIRING (1988) and CHEN (1994) have made use of this data set. The data consist of 25 observations of a process that has two quality characteristics of interest, namely, Brinell hardness (H) and tensile strength (S). The [USL, LSL] of ‘ H ’ were [241.3, 112.7] with target set at 177, while the specifications for ‘ S ’ were [73.30, 32.70] with $T = 53$. This suggests that both H and S were symmetric about their respective targets.

However, before assessing the capability of the process, we need to check the validity of the assumption of multivariate normality of the present data. While testing the multivariate normality of the data, it is observed that the p -value associated with Royston’s test is 0.02586 (using the ‘royston’ and ‘MVN’ packages of the open-source software R). As the p -value is less than 0.05, it is logical to expect that the underlying distribution of the present data set is not multivariate normal.

In order to assess the capability of the process, we now transform the data using the Box–Cox power transformation (Box and Cox, 1964). For multivariate Box–Cox power transformation, we have used the package ‘car’ of the open-source software R. The vector of transformation parameters is found to be $\lambda = (1, 2)$ with the corresponding p -value for the associated likelihood ratio test being 0.2250. The transformed data are given in Table 4.

The plotted data with the modified specification region, the modified target, and the elliptical process region are given in Figure 2.

The p -value corresponding to Royston’s multivariate normality test is 0.1103. Therefore, it is logical to expect that the transformed data set in Table 4, indeed, follows multivariate normal distribution.

Moreover, as the data set has been transformed to have multivariate normal distribution, it is now required to transform USL, LSL, and T , by virtue of the same transformation. The transformed specification limits and targets for H_{new} and S_{new} are $\text{USL}_{H_{\text{new}}} = 240.3$, $\text{LSL}_{H_{\text{new}}} = 111.7$, $T_{H_{\text{new}}} = 176$, $\text{USL}_{S_{\text{new}}} = 2685.945$, $\text{LSL}_{S_{\text{new}}} = 534.145$, and $T_{S_{\text{new}}} = 1404$. Also, the correlation coefficient between H_{new} and S_{new} is 0.846.

Table 4. Transformed data set

Variable	Sample no.									
	1	2	3	4	5	6	7	8	9	10
H_{new}	143	200	160	181	148	178	162	215	161	141
S_{new}	584.32	1624	1127.625	1425.82	1141.92	1325.625	1052.905	1745.905	1170.78	1118.145
H_{new}	11	12	13	14	15	16	17	18	19	20
S_{new}	175	187	187	186	172	182	177	204	178	196
	1641.145	1710.625	1693.12	1624	1219.68	1635.42	1279.68	1517.505	1294.905	1675.705
H_{new}	21	22	23	24	25	—	—	—	—	—
S_{new}	160	183	179	194	181	—	—	—	—	—
	1034.625	1452.105	1310.22	1652.625	1545.18	—	—	—	—	—

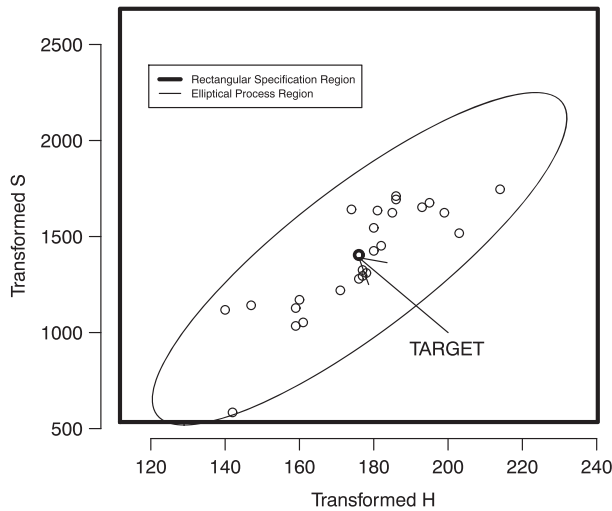


Fig. 2. Plot for transformed version of SULTAN's (1986) data.

Thus, although, apparently the specification region was symmetric with respect to the target vector, the transformed specification region is asymmetric about the transformed target vector, namely, $T_{\text{new}} = (T_{H_{\text{new}}}, T_{S_{\text{new}}}) = (177, 1404)$. This is similar to the observation by BOYLES (1994) regarding possible reasons of asymmetry in specification limits.

Based on the data in Table 4, $\mathbf{d} = (64.3, 1075.9)'$, $\mathbf{d}^* = (64.3, 869.855)'$, and $A = \begin{pmatrix} 0.0031 & 0 \\ 0 & 0.0228 \end{pmatrix}$. Thus, $G = A\mathbf{d} = (0.2, 24.5868)$ and $G^* = A\mathbf{d}^* = (0.2, 19.8782)$.

Also, $\hat{\boldsymbol{\mu}} = (176.20, 1384.122)'$, $\hat{\Sigma} = \begin{pmatrix} 338 & 4435.277 \\ 4435.277 & 81,311.074 \end{pmatrix}$.

Hence $\hat{C}_M(0, 0) = 1.1672$, $\hat{C}_M(1, 0) = 1.1623$, $\hat{C}_M(0, 1) = 1.1551$, and $\hat{C}_M(1, 1) = 1.1503$. Also, the threshold value of $\hat{C}_M(0, 0)$ is computed as 1.1672. Thus, the process is potentially just capable as the threshold value coincides with the value of $\hat{C}_M(0, 0)$, and this is supported by Figure 2 as well. All the other MPCIs are found to have values less than the threshold value. This connotes that the actual capability level of the process is not satisfactory.

Also, the close values of $\hat{C}_M(0, 0)$ and $\hat{C}_M(1, 0)$ suggest that the process suffers more owing to lack of proximity towards the target than owing to the prevailing level of dispersion. This observation is justified by the data itself. The values of S_{new} vary between 584.320 and 1745.905 with the target being 1404, while the range of the values of H_{new} is 140 to 214 and the target is 176. The wide range of S_{new} values strongly suggests that the process suffers from severe off-centeredness corresponding to this quality characteristic. Although owing to the high correlation between H_{new} and S_{new} , it will not be wise to conclude that the off-centeredness of the overall process is contributed by S_{new} only. Therefore, the process centering needs immediate attention of the concerned authority.

Moreover, one needs to take a look on the prevailing dispersion scenario of the process as well. Because, despite the $\hat{C}_M(1, 0)$ value being close to the $\hat{C}_M(0, 0)$ value, it is still less than the threshold value, indicating a high degree of dispersion of the quality characteristics. Also, as can be seen in Figure 2, the most lower part of the elliptical process region lies slightly outside the rectangular specification region, which also indicates high degree of asymmetry of the quality characteristics. Thus, $C_M(u, v)$ correctly assesses the capability of the process under consideration, by concluding that it is not capable of performing satisfactorily.

In this context, the estimated values of the variances of H_{new} and S_{new} (viz., 338 and 81,311.074, respectively) are widely apart; we have also standardized the data given in Table 4, following TANO and VÄNNMAN (2013). For the transformed data, we have $C_M(0, 0) = 1.167224$, $C_M(1, 0) = 1.162525$, $C_M(0, 1) = 1.15971$, and $C_M(1, 1) = 1.155041$ with the threshold value being 1.67224. Thus the results for the standardized and non-standardized data set almost coincide and yield identical interpretation about the health of the process.

SULTAN'S (1986) data have been used by many authors several times in order to compare the performances of several MPCIs (CHAN *et al.*, 1991; CHEN, 1994; and POLANSKY, 2001, among others). Among them, CHAN *et al.* (1991) and CHEN (1994) have assumed multivariate normality of the data and have concluded that the process is capable. On the contrary, POLANSKY (2001) has addressed the problem from non-parametric view point and has shown using kernel estimation procedure that the process is actually poor in terms of capability. According to POLANSKY (2001), such incapability of the process is due to a point that is near the boundary of the specification set. This is evident from Figure 2, featuring the plot of the transformed data as well. Thus, our view regarding the capability of the process is concurred by POLANSKY'S (2001) observation. This is because in all the cases except the case of POLANSKY (2001), the underlying distribution of the quality characteristics was assumed to be multivariate normal, while actually this is not the case. Consequently, the process was wrongly considered as capable, while in reality, it is actually far from being so. This strongly argues for checking the underlying distribution of the quality characteristics before proceeding with the capability assessment of a process.

Moreover, our approach of transforming the data to multivariate normality and then applying $C_M(u, v)$ is easier to execute as compared with using POLANSKY'S (2001) MPCI. Also, the salient features of $C_M(u, v)$, like the threshold value and acknowledging asymmetry of the loss function, makes it easier to interpret than POLANSKY'S (2001) MPCI and thereby fulfills our objective of defining user-friendly MPCIs for assessing capability of processes with more than one correlated quality characteristics and asymmetric specification region.

6 Concluding remarks

In the present article, we have discussed some very crucial as well as highly desirable properties of a superstructure of MPCIs $C_M(u, v)$ for asymmetric specification

region. The superstructure is designed in such a manner that by varying u and v , one can make a trade-off between process centering and proportion of non-conformance depending upon the situation. We have also derived the expression for threshold value of $C_M(0, 0)$. This helps in the decision-making procedure regarding the quality of the process. Moreover, this superstructure acknowledges asymmetry in the loss function. The estimation procedures for the plug-in estimators of the member indices of $C_M(u, v)$ have been discussed in detail. A real-life example has been provided to strengthen the theory developed in this article.

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