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Impact of multivariate normality assumption on multivariate process capability indices

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ABSTRACT

Process Capability Index (PCI) is a very popular tool for assessing performance of processes (often involving a single quality characteristic). Multivariate process capability indices (MPCI) are comparatively new to the literature and hence often involve some difficulties in practical applications. One such hurdle is multivariate normality assumption of the underlying distribution of the quality characteristics. While such assumption gives some computational as well as theoretical advantage in formulating MPCIs, often data encountered in practice do not follow multivariate normal distribution due to several reasons. Consequently, the computed values of the MPCIs may give misleading results. In the present article, we have made performance analysis of some MPCIs in the light of a dataset which has been widely used in the literature, particularly in the context of MPCIs. Most of these MPCIs were already applied to the said data in the literature and our objective is to make their comparative performance analysis. In this context, the data, though actually non-normal, is often concluded as multivariate normal by several researchers. Therefore, while making the performance analysis of the MPCIs, this aspect has also been incorporated to put emphasis on the importance of distributional assumption in a multivariate process capability analysis.

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Capability Index; multivariate non-normal distribution; multivariate normal distribution; multivariate process quality control; test for multivariate normality

1. Introduction

Process Capability Index (PCI) is a very important yardstick for assessing the overall performance of a process. Initially, PCIs were defined assuming that the process under study has only one quality characteristic of interest. Under the assumption of normality of the distribution of the quality characteristic under consideration, the four classical PCIs for univariate processes with bilateral specification limits are

$$C_p = \frac{USL - LSL}{6\sigma}, \tag{1}$$

$$C_{pk} = \frac{d - |\mu - M|}{3\sigma}, \tag{2}$$

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$$C_{pm} = \frac{d}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \tag{3}$$

$$C_{pmk} = \frac{d - |\mu - M|}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \tag{4}$$

where USL and LSL are, respectively, the upper and lower specification limits of a process, $d = \frac{USL-LSL}{2}$, $M = \frac{USL+LSL}{2}$ and T is the target of the process (refer to Kotz and Johnson 1992-2000; Pearn and Kotz 2007; Chakraborty and Chatterjee 2016; Chatterjee and Chakraborty 2014 and the references there-in).

Although quality characteristics with symmetric (with respect to the stipulated target) bilateral specification limits are mostly encountered in practice, those with asymmetric specification limits are also not rare in industries (refer to Boyles 1994). Chen and Pearn (2001) defined a super-structure of PCIs for asymmetric specification limits which is defined as

$$C_p^{''}(u, v) = \frac{d^* - uF^*}{3\sqrt{\sigma^2 + vF^2}}, \tag{5}$$

where $F = \max\{\frac{d^*(\mu-T)}{d_u}, \frac{d^*(T-\mu)}{d_l}\}$ and $F = \max\{\frac{d(\mu-T)}{d_u}, \frac{d(T-\mu)}{d_l}\}$. For $u = 0, 1$ and $v = 0, 1$, $C_p^{''}(0,0) = C_p^{''}$, $C_p^{''}(1,0) = C_{pk}^{''}$, $C_p^{''}(0,1) = C_{pm}^{''}$, and $C_p^{''}(1,1) = C_{pmk}^{''}$ which are analogous to C_p, C_{pk}, C_{pm}, and C_{pmk} defined in Eq. (4).

However, with the increasing use of modern cutting-edge technologies in industries, the concept of univariate processes (i.e., the processes producing items having a single important quality characteristic) is becoming rare day by day. Now a days, often a production engineer has to deal with more than one inter-related quality characteristics at a time. This necessitates use of the so-called multivariate process capability indices (MPCI).

Although univariate PCIs grossly out-number MPCIs in the literature (refer Pearn and Kotz (2007) and the references there-in), the area is attracting more and more eminent statisticians day by day due to its huge potential application.

One of the biggest challenges faced by the researchers in this field is the scarcity of data. Often it becomes difficult to obtain raw data on a single quality characteristic due to several reasons including ignorance among the concerned authority as well as the shop-floor people about the importance of such data, faulty data collection and storage system and so on. The situation worsens for multi-dimensional processes, as then one has to collect data on more than one quality characteristics at a time, so as to capture the inter-dependence of those quality characteristics. On the other hand, one cannot deny importance of such data in validating a new model in the context of multivariate process capability analysis. The only option left to the researchers is to reuse the data already available in the literature.

Sultan's (1986) data are a very popular dataset, which has been used time and again to study the performance of several MPCIs. However, most of these MPCIs are defined under some distributional assumptions, which need to be adhered for their successful applications. Hence, prior to the use of a dataset, one should check its compatibility with the MPCI under consideration as otherwise it may give totally misleading results.

In this context, while studying effect of testing normality in the context of estimating univariate process capability indices, Han (2006) has observed that in case the underlying distribution of the process is not known before hand, one should use test(s) for normality to ressolve the problem before proceeding to select appropriate PCI. This is true for multivariate processes as well.

Interestingly, in the context of univariate process capability analysis, study on impact of (univariate) normality assumption are widely available (refer to Han 2006; Pearn and Kotz 2007 and the references there-in). However, to the best of our knowledge, from the perspective of multivariate processes, such study is yet to be carried out.

In the present article, we have revisited Sultan's (1986) data and have critically studied the performance of some MPCIs, particularly from the view point of distributional assumptions of those MPCIs. We have primarily selected those MPCIs, on which Sultan's (1986) data have already been applied in the literature. This helps in easy understanding of impact of multivariate normality assumption on respective MPCIs, in a sense that, applying an MPCI, which is based on multivariate normality assumption, on a multivariate non-normal data, is very likely to yield misleading result.

In Sec. 2, a brief description is given regarding Sultan's (1986) data. Section 3 critically examines some MPCIs where these data were used, in the light of multivariate normality assumption followed by a concluding remark in Sec. 4.

2. On Sultan's (1986) dataset

Sultan's data are one of the most widely used datasets in the context of process capability analysis. The data were originally published by Sultan (1986) in the context of constructing acceptance charts for two correlated quality characteristics. The complete dataset is given in Table 1.

The plotted data with the rectangular specification region and the target are given in Figure 1.

The data in Table 1 consist of 25 observations of a process that has two correlated quality characteristics of interest, namely brinell hardness (H) and tensile strength (S).

For H, the specifications are $(USL_H, T_H, LSL_H) = (241.3, 177, 112.7)$, while for S, the specifications are $(USL_S, T_S, LSL_S) = (73.30, 53, 32.70)$, where, USL_X, T_X, and LSL_X stand for the upper specification limit (USL), target (T), and lower specification limit (LSL) of a particular quality characteristic, say X.

Table 1. Dataset pertaining to a manufacturing industry with two correlated quality characteristics and 25 randomly selected sample observations (refer to Sultan 1986).

	Sample No.									
Characteristic	1	2	3	4	5	6	7	8	9	10
Brinell Hardness (H)	143	200	160	181	148	178	162	215	161	141
Tensile Strength (S)	34.2	57.0	47.5	53.4	47.8	51.5	45.9	59.1	48.4	47.3
	Sample No.									
Characteristic	11	12	13	14	15	16	17	18	19	20
Brinell Hardness (H)	175	187	187	186	172	182	177	204	178	196
Tensile Strength (S)	57.3	58.5	58.2	57.0	49.4	57.2	50.6	55.1	50.9	57.9
	Sample No.									
Characteristic	21	22	23	24	25	–	–	–	–	–
Brinell Hardness (H)	160	183	179	194	181	–	–	–	–	–
Tensile Strength (S)	45.5	53.9	51.2	57.5	55.6	–	–	–	–	–

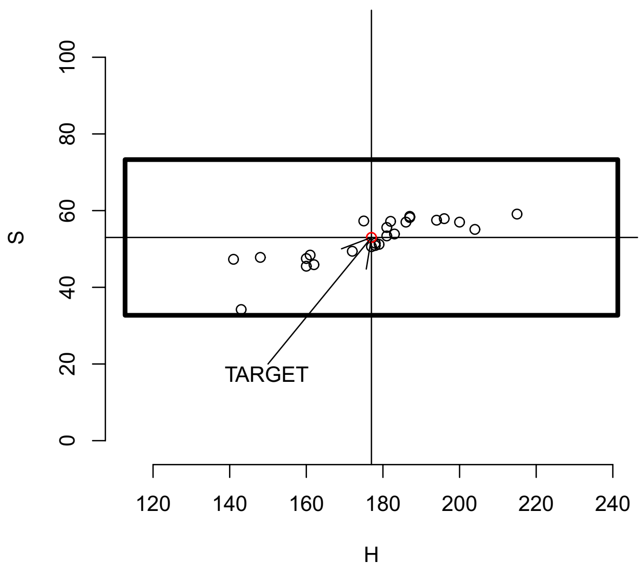

Figure 1. Plot for Sultan's (1986) data.

This suggests that, both "H" and "S" are symmetric about their respective targets. Moreover, since, based upon the data given in Table 1, the correlation coefficient between "H" and "S" is 0.834, the capability assessment of the process with respect to individual quality characteristics is not advocated—rather, suitable MPCI should be used for the said purpose.

Although Sultan's data (1986) were originally used for acceptance chart, Chen (1994) first used the data in the context of multivariate process capability analysis and due to the scarcity of suitable multivariate data, gradually these data became very popular among the statisticians and quality engineers, who used this for assessing the performance of their newly proposed MPCIs.

Most of these MPCIs are defined based on the assumption that the underlying distribution of the quality characteristics under consideration is multivariate normal (refer to Chen 1994; Shahriari 2009; Pan 2010; Wang 1998; Wang 2000 and so on). However, one should test the authenticity of such strong distributional assumption rigorously before actually using this, because the inference regarding the actual health of a process may largely depend on such assumption (refer to Wang 2000).

Therefore, before using a dataset, Sultan's (1986) dataset in particular, in a multivariate process capability analysis, we need to check the validity of the assumption of multivariate normality of the present data. We use R software for this purpose and observe that

1. The p-value associated with Shapiro–Wilk test is 0.006764,
2. The p-value associated with Royston's test is 0.02586.

Since both these p-values are less than 0.05, it is logical to expect that the underlying distribution of the present dataset is not multivariate normal.

Interestingly, if we still ignore the correlation between H and S, individually, H is found to be normal with the p-value for Shapiro–Wilk normality test being 0.6271, while S is non-normal with p-value 0.007877.

Hence, the capability of the process can be assessed through either of the following two approaches:

1. Transform the original data into multivariate normal and then apply suitable MPCIs;
2. Directly apply MPCIs designed for multivariate non-normal data.

For transforming the data into a multivariate normal one, multivariate Box–Cox method will be applied here. This method uses separate transformation parameter for each variable, which is very relevant in our case as here one variable (H) is normal and the other one is non-normal. Moreover, here classification of the variables in dependent/independent form is not required. When the variables are finally transformed into joint normality, they become approximately linearly related, constant in conditional variance and marginally normal in distribution.

We have used the packages "mvnormtest" and "MVN" of the software R for the purpose of the said transformation. Here, the vector of transformation parameter is $\lambda = (1, 2)$. The transformed data are given in Table 2.

Note that, since "H" was already having a normal distribution, its values remain unchanged; while, for "S" the transformed values have significantly increased. The correlation coefficient between H_{new} and S_{new} is 0.846, which is slightly more than that between the original variables.

Moreover, since the dataset has been transformed to have multivariate normal distribution, it is now required to transform **USL**, **LSL**, and **T**, by virtue of the

Table 2. Transformed dataset corresponding to Table 1.

	Sample no.									
Variable	1	2	3	4	5	6	7	8	9	10
H_{new}	143	200	160	181	148	178	162	215	161	141
S_{new}	584.32	1,624	1,127.625	1,425.82	1,141.92	1,325.625	1,052.905	1,745.905	1,170.78	1,118.145
	Sample no.									
Variable	11	12	13	14	15	16	17	18	19	20
H_{new}	175	187	187	186	172	182	177	204	178	196
S_{new}	1,641.145	1,710.625	1,693.12	1,624	1,219.68	1,635.42	1,279.68	1,517.505	1,294.905	1,675.705
	Sample no.									
Variable	21	22	23	24	25	–	–	–	–	–
H_{new}	160	183	179	194	181	–	–	–	–	–
S_{new}	1,034.625	1,452.105	1,310.22	1,652.625	1,545.18	–	–	–	–	–

same transformation. The transformed specification limits and targets for H_{new} and S_{new} are as follows:

$$\left.\begin{array}{l} USL_{H_{new}} = 240.3 \\ LSL_{H_{new}} = 111.7 \\ T_{H_{new}} = 176 \end{array}\right\} \qquad \left.\begin{array}{l} USL_{S_{new}} = 2685.945 \\ LSL_{S_{new}} = 534.145 \\ T_{S_{new}} = 1404.000 \end{array}\right\}$$

Thus, although, apparently the specification region was symmetric with respect to the target vector, the transformed specification region is asymmetric about the transformed target vector, namely $T_{new} = (T_{H_{new}}, T_{S_{new}})' = (177, 1404)'$. This is similar to the observation by Boyles (1994) regarding possible reasons of asymmetry in specification limits.

The plotted data with the modified specification region, the modified target, and the elliptical process region are given in Figure 2.

The p-values for the Shapiro–Wilk multivariate normality test and Royston's test are 0.07627 and 0.1103, respectively, indicating toward the successful transformation of the data into multivariate normal. Therefore, the MPCIs (with multivariate normality assumption), for which initially Sultan's (1986) original data were used, need to be recomputed for the transformed data given in Table 2.

In this context, from Figure 2, it is evident that the sample observations are highly scattered from the target. Even, the elliptical process region is slightly outside the rectangular specification region. Therefore, it is logical to conclude that the process is not performing satisfactorily.

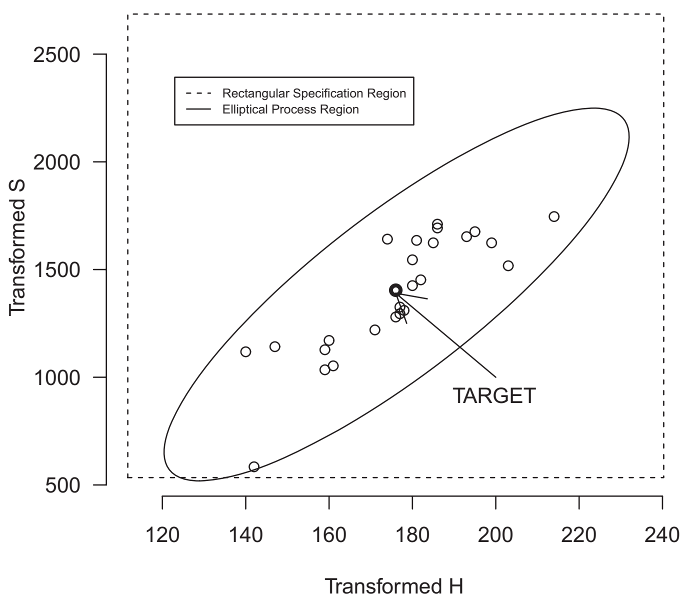

Figure 2. Plot for Sultan's (1986) transformed data.

3. Revisiting some MPCIs which used Sultan's (1986) data

We shall now review some of the MPCIs which used Sultan's data and shall make a performance analysis of those indices in light of the distributional assumption of the concerned quality characteristics. For this, we shall divide the MPCIs under consideration into following classes:

1. MPCIs using a principal component analysis.
2. MPCIs based on ratio of a process region and specification region.
3. MPCIs based on the concept of proportion of non-conformance.
4. Vector-valued MPCIs.

3.1. MPCIs using the principal component analysis

Principal component analysis (PCA) is used in process capability studies for dimension reduction which, in turn, often enables us to adopt simpler formulation. For example, Sultan's (1986) data are originally a bivariate one. However, after application of PCA, one can retain a single principal component (PC) with a very insignificant loss of variation (0.12%) and as a result, univariate PCIs can be applied to the first PC to assess capability of the process.

3.1.1. MPCIs given by Wang and Chen (1998)

Wang and Chen (1998) first used the concept of the principal component analysis (PCA) in the context of a process capability analysis. Under the assumption of multivariate normality of data, they defined MPCIs MC_p, MC_{pk}, MC_{pm}, and MC_{pmk} analogous to C_p, C_{pk}, C_{pm}, and C_{pmk} [see Eq. (4)], respectively, as

$$MC_p = \left(\prod_{i=1}^{v} C_{p:PC_i}\right)^{1/v}, \tag{6}$$

$$MC_{pk} = \left(\prod_{i=1}^{v} C_{pk:PC_i}\right)^{1/v}, \tag{7}$$

$$MC_{pm} = \left(\prod_{i=1}^{v} C_{pm:PC_i}\right)^{1/v}, \tag{8}$$

$$MC_{pmk} = \left(\prod_{i=1}^{v} C_{pmk:PC_i}\right)^{1/v}. \tag{9}$$

Here, v ($\leq$ q) is the number of principal components which are retained. Also, $C_{p:PC_i}$, $C_{pk:PC_i}$, $C_{pm:PC_i}$, and $C_{pmk:PC_i}$ represent, respectively, the univariate measures of PCIs viz. C_p, C_{pk}, C_{pm}, and C_{pmk} for the ith principal component, for $i = 1(1)v$.

Based on Sultan's data, Wang and Chen (1998) computed $MC_p = MC_{pk} = MC_{pm} = MC_{pmk} = 1.18$. Hence, they concluded that the process is performing satisfactorily with the process mean being close to the stipulated target.

However, since for Sultan's (1986) data, the assumption of multivariate normality is violated, generalization of PCIs like C_p and so on, which are defined under the said assumption, are not applicable here. Hence, we need to reassess the capability level of the process afresh.

In this context, multivariate normality assumption is not mandatory for PCA. However, in the subsequent step, i.e., while selecting appropriate MPCI, one needs to pay attention to the respective distributional assumption.

For the transformed data in Table 2, the vector of eigen values is $\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 81553.2905 \\ 95.7835 \end{pmatrix}$. Also, the two eigen vectors are $\boldsymbol{u_1} = \begin{pmatrix} 0.05453 \\ 0.9985 \end{pmatrix}$ and $\boldsymbol{u_2} = \begin{pmatrix} -0.9985 \\ 0.0545 \end{pmatrix}$.

Here, we have denoted the vectors through bold-faced letters.

Thus, variation explained by the first principal component (PC) is $\frac{\lambda_1}{\lambda_1+\lambda_2} = 0.9988$, i.e., 99.88% of the total variation and that by the second PC is $\frac{\lambda_2}{\lambda_1+\lambda_2} = 0.0012$, i.e., 0.12% of the total variation. Therefore, it is sufficient to retain the first PC only.

As has already been discussed in Sec. 2, for the transformed data, the specification region has become asymmetric with respect to the target and hence the PCIs given in Eq. (4) will no more be applicable here. Rather, one has to use the PCIs given in Eq. (5) to assess the capability of the process.

For the first principal component corresponding to the data in Table 2, $MC''_p = C''_{p:PC_1} = 1.017906$, $MC''_{pk} = C''_{pk:PC_1} = 0.9947$, $MC''_{pm} = C''_{pm:PC_1} = 1.0177$, and $MC''_{pmk} = C''_{pmk:PC_1} = 0.9946$.

Now, following Chatterjee and Chakraborty (2014) the threshold value of $C''_{p:PC_1}$ is 0.80911, which means the process is potentially capable. The values of all the other MPCIs are also larger than this threshold value, indicating towards the satisfactory performance of the process. However, as has already been observed, the process, as represented by the transformed data (vide Figure 2), is not performing satisfactorily and the MPCIs given by Wang and Chen (1998) fail to capture this phenomenon.

3.1.2. Tano and Vannman's (2013) MPCI

Tano and Vannman (2013) have proposed an MPCI based only on the first principal component. The authors observed that, although the variability with respect to a quality characteristic may apparently look like moderate or small, the same amount of variability, when considered with respect to the respective specification limits, may be quite high and may even exceed the specification span. This made the authors standardize the variables before applying the so-called PCA.

If there exists q correlated quality characteristics represented by q random variables $X_1, X_2, \ldots, X_q$, then for the ith random variable, the authors suggested the following standardization:

$$X_i^{(T)} = \frac{X_i - M_i}{d_i}, \quad \forall i = 1(1)q, \tag{10}$$

where $M_i = \frac{U_i + L_i}{2}$ and $d_i = \frac{U_i - L_i}{2}$.

Let $\lambda_1^{(T)}$ and $\boldsymbol{U}_1^{(T)}$ denote the first eigen value and the first eigen vector, respectively, and $PC_1^{(T)}$ denotes the first principal component corresponding to the standardized data. Tano and Vannman (2013) have formulated the upper and lower specification limits of $PC_1^{(T)}$ as $USL_{PC_1^{(T)}} = \frac{1}{\max\limits_i |U_{1i}^{(T)}|}$ and $LSL_{PC_1^{(T)}} = -\frac{1}{\max\limits_i |U_{1i}^{(T)}|}$, respectively, where, $\max\limits_i |U_{1i}^{(T)}|$ is the maximum absolute value of the components of $\boldsymbol{U_1}^{(T)}$.

Then based on $PC_1^{(T)}$ and analogous to C_p, Tano and Vannman (2013) defined a new MPCI as

$$C_{p,TV} = \frac{1}{3\sqrt{\lambda_1^{(T)}} \times \{\max\limits_i |U_{1i}^{(T)}|\}}, \tag{11}$$

Tano and Vannman (2013) used the original dataset given in Table 1 and computed $C_{p,TV} = 1.216$, a threshold value $k_0 = 1.1124$ and the value of a lower confidence bound (LCB) as $0.924 < k_0$. Hence, the process is not performing satisfactorily.

Let us now observe the performance of $C_{p,TV}$ in the context of Sultan's transformed data given in Table 2.

Following Tano and Vannman (2013), we first standardize the data in Table 2. For these standardized data, the mean vector is $\boldsymbol{\mu}^{(T)} = \begin{pmatrix} 0.0031 \\ -0.21 \end{pmatrix}$ and the dispersion matrix is $\Sigma^{(T)} = \begin{pmatrix} 0.0817 & 0.0641 \\ 0.0641 & 0.0702 \end{pmatrix}$.

The first eigen value and the first eigen vector for the standardized data are $\lambda_1^{(T)} = 0.1404$ and $\boldsymbol{u}_1^{(T)} = \begin{pmatrix} -0.7380 \\ -0.6748 \end{pmatrix}$.

Therefore, $\max\limits_{i=1,2} |\hat{u}_{1i}^{(T)}| = 0.7380$ and hence, $USL_{PC_1^{(T)}} = 1/0.7380 = 1.3549$ and $LSL_{PC_1^{(T)}} = -1.3549$.

Thus, from Eq. (11), $\hat{C}_{p,TV} = 1.2054$. Although, apparently, this indicates toward a satisfactory process performance, a deeper insight into the process will reveal a completely different picture. Following Tano and Vannman (2013), the lower confidence bound (LCB) for $C_{p,TV}$ is 0.9157, while the threshold value is $k_0 = 1.0809$ which exceeds the LCB value and hence the process is not capable.

In this context, Tano and Vannman (2013) did not standardize the target vector and consequently, did not check whether the specification limits are symmetric or not.

For the normalized data of Table 2, the target vector, after standardization, becomes $\boldsymbol{T}^{(T)} = (0, -0.1915)'$. Hence for the first principal component, following Wang and Chen (1998), the target vector will be $T_{PC_1^{(T)}} = \hat{\boldsymbol{u}}_1{}^{(T)\prime}\boldsymbol{T}^{(T)} = 0.1292 \neq \frac{LSL_{PC_1^{(T)}} + USL_{PC_1^{(T)}}}{2} = 0$, which means that the specification region is asymmetric with respect to the standardized target vector and hence $C_{p,TV}$, which is defined analogous to C_p, will not be applicable here. Rather we have to define an MPCI,analogous to C_p'' as follows:

$$C_{p,TV}'' = \frac{d^*_{PC_1^{(T)}}}{3\sqrt{\lambda_1^{(T)}}}, \tag{12}$$

where $d^*_{PC_1^{(T)}} = \min\{d_{l,PC_1^{(T)}}, d_{u,PC_1^{(T)}}\}$, with $d_{l,PC_1^{(T)}} = T_{PC_1^{(T)}} - LSL_{PC_1^{(T)}}$ and $d_{u,PC_1^{(T)}} = USL_{PC_1^{(T)}} - T_{PC_1^{(T)}}$.

Thus, for transformed and standardized Sultan's (1986) data, $C_{p,TV}'' = 1.0905$.

Note that, since the concepts of threshold value and LCB were defined by Tano and Vannman (2013) for symmetric specification region, they will not be applicable in the present context. However, following Chatterjee and Chakraborty (2014), the proportion of nonconformance (PNC) associated with the $C_{p,TV}''$ value of 1.0905 is 0.1378. This implies, under the present production scenario, the maximum observable PNC will be 13.78 % and this is quite high. Therefore, the transformed and standardized data also indicate toward the unsatisfactory performance of the process. Infact, the process is not even potentially capable, as both $C_{p,TV}$ and $C_{p,TV}''$ measure potential capability of a process.

In this context, for the transformed and standardized data, only 92.35 % of the total variability is explained by the first principal component, while the remaining 7.65 % of the total variability still remain unexplained.

If the original data (in Table 1) is standardized, as is done by Tano and Vannman (2013), the explained variation by the first principal component will be only 91.69 %, i.e., the situation will be even worse.

3.1.3. Wang and Du's (2000) MPCI

In the context of the MPCIs based on PCA, Wang and Du (2000) have suggested the use of MPCIs given by Wang and Chen (1998) for multivariate normal processes, while for rest of the processes, they followed Luceno (1996) to define an MPCI, namely MC_{pc}, analogous to C_p, as follows:

$$MC_{pc} = (\prod_{i=1}^{\nu} C_{pc;PC_i})^{1/\nu}, \tag{13}$$

where ν is the number of significant principal components, $C_{pc;PC_i}$ is the value of $C_{pc} = \frac{USL_i - LSL_i}{6\sqrt{\frac{\pi}{2}}E|Y_i - M|}$ (vide Luceno 1996) corresponding to the ith PC, say Y_i, for $i = 1(1)\nu$ and USL_i, LSL_i, and M_i are, respectively, the USL, LSL and specification mid-point for the ith PC, for $i = 1(1)m$.

In this context, the authors, assuming the process corresponding to the data in Table 1 as a multivariate normal one, used the MPCIs given by Wang and Chen (1998) and hence concluded that the process is performing satisfactorily.

We used the original data as given in Table 1 and since it is not following multivariate normal distribution, we have used Luceno's (1996) index given in Eq. (13) for the purpose of process capability analysis and have observed that

$MC_{pc} = 1.245013$. Also, following Luceno (1996), the 95% confidence interval for MC_{pc} is (1.9088, 2.5910). Since the observed value of MC_{pc} is smaller than the lower confidence limit, one can conclude, with 95% confidence, that the process is not even potentially capable and hence there is no point in computing other MPCI values analogous to C_{pk}, C_{pm}, and C_{pmk}.

Therefore, although Wang and Du (2000) originally considered the process to be capable, application of their MPCI viz. MC_{pc} to Sultan's (1986) transformed data clearly indicated toward the poor performance of the process.

3.2. MPCIs defined as the ratio of the specification region and the process region

3.2.1. MPCIs Given by Taam, Subbaiah, and Liddy (1993)

The MPCIs given by Taam, Subbaiah, and Liddy (1993) are among the oldest MPCIs in the literature of statistical quality control. The authors defined an MPCI analogous to C_p as

$$\begin{aligned} MC_p &= \frac{\text{Volume of modified tolerance region}}{\text{Volume of the 99.73\% process region}} \\ &= |\rho^*|^{-\frac{1}{2}}, \quad \text{(refer Pan and Lee (2010))}. \end{aligned} \tag{14}$$

where ρ^* is the correlation matrix corresponding to the quality characteristics under consideration.

Again, analogous to C_{pm}, Taam, Subbaiah, and Liddy (1993) defined MC_{pm} as,

$$\begin{aligned} MC_{pm} &= \frac{\text{Vol. (Modified tolerance region)}}{Vol.\ ((\boldsymbol{X} - \boldsymbol{\mu})'\Sigma_T^{-1}(\boldsymbol{X} - \boldsymbol{\mu}) \leq k(q))} \\ &= MC_p \times D^{-1} \end{aligned} \tag{15}$$

where $\boldsymbol{X} \sim N_q(\boldsymbol{\mu}, \Sigma)$, $\Sigma_T = E[(\boldsymbol{X}-\boldsymbol{T})(\boldsymbol{X}-\boldsymbol{T})']$, $k(q)$ is the 99.73th percentile of χ^2 distribution with q degrees of freedom and $D = \{1+(\boldsymbol{x}-\boldsymbol{\mu})'\Sigma_T^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\}^{\frac{1}{2}}$.

Although Taam, Subbaiah, and Liddy (1993) themselves did not apply Sultan's (1986) data to their MPCIs, performance of their MPCIs has been analyzed and compared with a number of other MPCIs in the literature, at different points in time. Moreover, these are among the most famous MPCIs among the practitioners. Hence, it is worthy to investigate the performance of MC_p and MC_{pm} in the light of the assumption of multivariate normality of the underling process distribution.

Now, since in the present context, there are only two quality characteristics, ρ^* will boil down to ρ. For the data in Table 2, $\rho = 0.8460$ and $D = 1.010954$. Hence, $MC_p = 1.8757$ and $MC_{pm} = 1.8533$.

Interestingly, the value of MC_p for the transformed data is very close to that for untransformed data (1.88). This is due to the fact that, following Pan and Lee (2010), MC_p can be expressed as a function of ρ only; while in Box–Cox transformation for multivariate normality, the variables are transformed in such a way that the original correlation structure is retained as much as possible.

Both of the values of MC_p and MC_{pm} are quite high, indicating toward the satisfactory performance of the process, which contradicts our initial observation that the process is not capable and hence following Pan and Lee (2010), we may say that MC_p and MC_{pm} over-estimate process capability.

The evaluation of process performance based on several MPCIs, which we have discussed so far, are based on sample observations and hence are subject to sampling fluctuations of different degrees. Therefore, instead of considering the MPCI values computed from the available data as the actual MPCI values, one should consider them as the estimated MPCI values and the properties of such estimators should be studied extensively. Pearn, Wang, and Yen (2007) have precisely done this job for the MPCIs defined by Taam, Subbaiah, and Liddy (1993).

For the normally transformed data in Table 2, following Pearn, Wang, and Yen (2007), the value of the plug-in estimator (i.e., the estimator obtained merely by replacing $\boldsymbol{\mu}$ and Σ by their sample counterparts) of MC_p, say $\hat{MC}_p = 2.0925$ and $\hat{MC}_{pm} = 2.0698$. Also, the unbiasedly estimated value of MC_p is 1.9181 and the 95% confidence interval for the value of MC_p is $[1.2712, 2.9041]$. Finally, similar to Pearn, Wang, and Yen (2007), the following set of hypotheses can be designed to test whether MC_p value is below the conventional threshold, namely 1: $H_0 : MC_p \leq 1$ against $H_1 : MC_p > 1$.

At the 5% level of significance, the critical value of the test is $c = 1.316453$ and since $\hat{MC}_p > c$, it is logical to expect that the true value of MC_p is greater than 1 at the 95% confidence level and hence the process can be considered as capable.

However, such optimistic conclusion about the process should be taken with a pinch of salt because, although, the general convention is to consider 1 as the threshold value for a PCI, it is actually ideal for the univariate ones and to be more precise, for C_p (refer to Chatterjee and Chakraborty 2014, 2015). Since the hypothesis testing is designed to test whether the observed MC_p value is greater than a particular value (in this case, 1) or not, it may not be able to predict the true health of the process, unless a thorough study is carried out regarding the threshold value of MC_p.

The problem of measurement error is one of the major obstacles for the accurate assessment of process capability. Shishebori and Hamadani (2008) have made a vivid discussion regarding the impact of gauge measurement errors in the context of the process capability analysis. In particular, they have considered MC_p for the purpose of mathematical formulation.

Suppose a random vector $\boldsymbol{M}$ denotes the measurement error under multivariate setup such that $\boldsymbol{M} \sim N_q(\mathbf{0}, \Sigma_{Me})$. Then, following Montgomery and Runger (2013), the gauge capability may be defined as

$$\lambda^M = \frac{(\pi \chi^2_{q,0.9973})^{q/2} \, |\Sigma_{Me}|^{1/2} [\Gamma(q/2+1)]^{-1}}{\text{Vol. (Modified Tolerance Region)}}. \tag{16}$$

Suppose $\boldsymbol{X}$ and $\boldsymbol{Y}$ are the random vectors denoting the true measures of the quality characteristics and the measures affected by the measurement error respectively, such that $\boldsymbol{Y} \sim N_q(\boldsymbol{\mu}, \Sigma_Y = \Sigma + \Sigma_{Me})$.

Shishebori and Hamadani (2008) have proposed a modification of MC_p for a process, for which the concerned measurements are contaminated by measurement error, as

$$MC_p^Y = \frac{MC_p}{\sqrt{(\lambda^M \, MC_p)^2 + \frac{|\Sigma_Y| + |\Sigma_{Me}|}{|\Sigma_Y - \Sigma_{Me}|}}} \tag{17}$$

The authors considered the original dataset given in Table 1 and assumed that the sample dispersion matrix for the gauge measurement error is $\hat{\Sigma}_{Me} = \begin{pmatrix} 33.8 & 0 \\ 0 & 22.5 \end{pmatrix}$.

For the transformed data in Table 2 and for the same $\hat{\Sigma}_{Me}$, $\hat{\lambda}^M = 5458.318$ and $\hat{MC_p}^Y = 23899.89$.

This highly exaggerates the true capability scenario of the process, which is contributed by the high value of $\hat{\lambda}^M$ triggered by high values of the specification limits of the second variable (tensile strength) after transformation into normality.

3.2.2. Pan and Lee's (2010) MPCIs

As has already been discussed, Pan and Lee (2010) have observed that, a major drawback of Taam, Subbaiah, and Liddy's (1993) MPCIs is the overestimation of process capability. To overcome this problem, they have proposed a new set of MPCIs as follows:

$$NMC_p = \left\{ \frac{|A^*|}{|\Sigma|} \right\}^{1/2}, \tag{18}$$

$$NMC_{pm} = \frac{NMC_p}{D}, \tag{19}$$

where A^* is a $q \times q$ matrix, whose $(i,j)^{th}$ element, namely a_{ij} is given by $a_{ij} = \rho_{ij} \left(\frac{U_i - L_i}{2\sqrt{\chi^2_{1-\alpha,q}}} \right) \times \left(\frac{U_j - L_j}{2\sqrt{\chi^2_{1-\alpha,q}}} \right)$ for $i, j = 1(1)q$.

For the original data in Table 1, the authors computed $NMC_p = 1.04$ and $NMC_{pm} = 1.01$ with the 95% confidence interval for both the MPCIs being [0.63, 1.44] and [0.63, 1.41], respectively. Therefore, the authors considered the process to be capable.

However, NMC_p and NMC_{pm} are defined assuming that the underlying process distribution is multivariate normal. Therefore, we shall now use the transformed data in Table 2 for recalculating the NMC_p and NMC_{pm} values.

For the transformed data, $A^* = \begin{pmatrix} 349.5213 & 4947.914 \\ 4947.914 & 97857.82 \end{pmatrix}$.

Hence, $NMC_p = 1.1156$ and $NMC_{pm} = 1.1035$. Also, the 95% confidence interval for both the MPCIs are [0.5596, 1.80] and [0.5865, 1.8392], respectively.

Therefore, although, Pan and Lee's (2010) MPCIs still consider the process to be capable, the amount of over-estimation is clearly more in case of Taam, Subbaiah, and Liddy's (1993) MPCIs. In other words, NMC_p and NMC_{pm} perform better than MC_p and MC_{pm}, though the result is still not completely satisfactory.

3.2.3. Goethals and Cho's (2011) MPCI

Goethals and Cho's (2011) have proposed the following MPCI, analogous to C_{pm}, which takes into account the fact that the loss incurred due to deviation of the process centering from the target may vary from one quality characteristic to another:

$$MC_{pmc} = \left[\frac{\prod_{i=1}^{q} (USL_i - LSL_i)}{\text{Vol. (99.73\% elliptical process region)}} \right] \times \frac{1}{D^G} = \frac{V_R}{D^G}, \text{ say} \tag{20}$$

where, Vol. (99.73% elliptical process region) $= \left\{ \pi \, \chi^2_{q,0.0027} \right\}^{q/2} \times [\Gamma(q/2 + 1)]^{-1} | \sum_{i=1}^{q} \sum_{j=1}^{i} \Sigma_{ij}|^{1/2}$ and where δ_{ii} and δ_{ij} denote, respectively, the loss coefficients associated with the ith quality characteristic and between the ith and jth quality characteristics.

For the transformed data given in Table 2, $V_R = 5.0378$.

Again, following Goethals and Cho (2011), we assume that the impact of quality loss due to deviation from target is the same for all quality characteristics and $\delta_{11} = \delta_{22} = 0.5$ and $\delta_{12} = 0$. Then $D^G = 1.0045$ and hence $MC_{pmc} = 5.015$.

Now, following Goethals and Cho (2011), the lower bound for V_R is $V_R^{LB} = 524.7667 > V_R$. This implies that the process suffers from high variability and hence is not performing satisfactorily.

In this context, a drawback of the MPCIs by Goethals and Cho (2011) is that, the MPCI value is still quite high, even when the process is incapable and this has been admitted by the authors themselves.

A minor modification in the definition of MC_{pmc} may be suggested in this regard. As can be observed from Eq. (20), V_R is defined analogous to C_p. Hence, similar to Taam, Subbaiah, and Liddy (1993), the denominator of V_R can be redefined as $\left\{\pi\ \chi^2_{q,0.0027}\right\}^{q/2} \times [\Gamma(q/2+1)]^{-1} \times |\Sigma|^{1/2}$. MC_{pmc} will get modified accordingly. Let us denote the redefined V_R and MC_{pmc} as V_R^* and MC_{pmc}^* respectively. Then for the transformed data, $V_R^* = 1.63226$ and $MC_{pmc}^* = 1.6249$.

The advantage of using this modification is that, the values of the MPCIs are now scaled down without compromising with their performances.

3.2.4. Chen's (1994) MPCI

Chen (1994) had first used Sultan's (1986) data in the context of the multivariate process capability analysis. The author had proposed a ratio-based MPCI which does not require the multivariate normality assumption and which is applicable to any type of tolerance zone.

Chen (1994) defined a general version of the tolerance zone as

$$V = \{\boldsymbol{x} \in \mathbb{R}^q : h(\boldsymbol{x} - \boldsymbol{T}) \leq r_0\} \tag{21}$$

where, $r_0 > 0$ and $h(\boldsymbol{x})$ is a positive homogeneous function, such that for any $t > 0$ and $\boldsymbol{x} \in \mathbb{R}^q$, $h(t\boldsymbol{x}) = th(\boldsymbol{x})$.

Then MC_p^{chen} can be defined as

$$MC_p^{chen} = \frac{r_0}{r}, \tag{22}$$

where, $r = \min\{c : P[h(\boldsymbol{x}-\boldsymbol{T}) \leq c] \geq 1-\alpha\}$. Usually, $\alpha = 0.0027$ is considered.

Applying MC_p^{chen} to the data in Table 2 and considering 3.5σ limits (similar to Chen (1994)), $MC_p^{chen} = 1/0.0925 = 1.108$.

Since, according to Chen (1994), MC_p^{chen} is a natural generalization of C_p, the process is likely to be potentially capable. Clearly, MC_p^{chen} overestimates process performance.

3.2.5. Chatterjee and Chakraborty's (2017) MPCI

Chatterjee and Chakraborty (2011; 2016) have defined a super-structure of MPCIs for asymmetric specification region as

$$C_M(u,v) = \frac{1}{3}\sqrt{\frac{(\mathbf{d}^* - u\mathbf{G}^*)'\Sigma^{-1}(\mathbf{d}^* - u\mathbf{G}^*)}{1 + v\mathbf{G}'\Sigma^{-1}\mathbf{G}}}, \tag{23}$$

where u and v are two nonnegative real numbers

$\boldsymbol{d}^* = (\min(d_{l1}, d_{u1}), \min(d_{l2}, d_{u2}), \ldots, \min(d_{lq}, d_{uq}))'$, i.e. $d_i^* = min(d_{li}, d_{ui})$, for $i = 1(1)q$ with $\mathbf{d}_u = (d_{u1}, d_{u2}, \ldots, d_{uq})'$ and $\mathbf{d}_l = (d_{l1}, d_{l2}, \cdot, d_{lq})'$, $d_{ui} = U_i - T_i$ and $d_{li} = T_i - L_i$ for $i = 1(1)q$.

Also, "$\mathbf{G}$" can be defined as $\mathbf{G} = (a_1 d_1, a_2 d_2, \ldots, a_q d_q)'$, where, $a_i = \left[\max\{\frac{\mu_i - T_i}{d_{ui}}, \frac{T_i - \mu_i}{d_{li}}\}\right]$ and $d_i = \frac{U_i - L_i}{2}$, for $i = 1(1)q$, such that, $A = \left[diag(a_1, a_2, \ldots, a_q)\right]$ is a $(q \times q)$ diagonal matrix and $\boldsymbol{d} = (d_1, d_2, \ldots, d_q)'$ is a 'q' - component vector with $d_i = \frac{U_i - L_i}{2}$. Thus, $\mathbf{G} = \left[diag(a_1, a_2, \ldots, a_q)\right] \times \boldsymbol{d} = A\mathbf{d}$.

Since $C_M(u,v)$ assumes the underlying process distribution to be multivariate normal, for data in Table 2, $\widehat{C}_M(0,0) = 1.1672$, $\widehat{C}_M(1,0) = 1.1623$, $\widehat{C}_M(0,1) = 1.1551$, and $\widehat{C}_M(1,1) = 1.1503$. Also, following Chatterjee and Chakraborty (2014), the threshold value of $\widehat{C}_M(0,0)$ is computed as 1.1672. Thus, the process is potentially just capable as the threshold value coincides with the value of $\widehat{C}_M(0,0)$ and this is supported by Figure 2 as well. All the other MPCIs are found to have values less than the threshold value.

In this context, we may recall that, threshold value is primarily computed for $C_M(0,0)$ and suggests the minimum value of $C_M(0,0)$ required for a process to be at least potentially capable. Since the other member indices of a super-structure like $C_M(u,v)$, with $u = 1$ and $v = 0, 1$, cannot assume higher value than $C_M(0,0)$, they are also desired to assume values exceeding the threshold value.

However, all of $\widehat{C}_M(1,0)$, $\widehat{C}_M(0,1)$, and $\widehat{C}_M(1,1)$ have values lower than the threshold value. This connotes that the actual capability level of the process is not satisfactory. Also, the close values of $\widehat{C}_M(0,0)$ and $\widehat{C}_M(1,0)$' and the fact that the value of $C_M(0,0)$ coincides with the threshold value, suggest that the process suffers more due to lack of proximity towards the target, than due to the prevailing level of dispersion.

This observation is justified by the data itself. The values of S_{new} varies between 584.320 and 1745.905 with the target being 1404; while the range of the values of H_{new} is 140 to 214 and the target is 176. The wide range of S_{new} values strongly suggests that the process suffers from severe off centeredness corresponding to this quality characteristic. Although, due to high correlation between H_{new} and S_{new}, it will not be wise to conclude that the off-centeredness of the overall process is contributed by S_{new} only.

Therefore, the process centering needs immediate attention of the concerned authority. Moreover, one needs to take a look on the prevailing dispersion scenario of the process as well. Because, despite the $\widehat{C}_M(1,0)$ value being close to the $\widehat{C}_M(0,0)$ value, it is still less than the threshold value, indicating high degree of dispersion of the quality characteristics.

Thus, $C_M(u,v)$ correctly assesses the capability of the process, by concluding that it is not capable of performing satisfactorily.

3.3. MPCIs based on the concept of proportion of nonconformance

Proportion of nonconformance (PNC) is a parallel approach of assessing process performance. It gives the proportion of items which do not conform to the preassigned specification limits, among all the produced items. Hence, bridging the two concepts, namely PCI and PNC have drawn the attention of many eminent statisticians since long (refer to Kotz and Johnson 1992-2000; Pearn and Kotz 2007 and the references there in).

3.3.1. Pearn, Wang, and Yen's (2006) MPCI

Pearn, Wang, and Yen (2006) have put emphasis on establishing relationship between MPCIs and the corresponding proportion of nonconformance (PNC). For this, the authors have defined a multivariate analogue of S_{pk}.

In this context, Boyles (1994) defined S_{pk}, for assessing capability of a univariate process with asymmetric specification limits as

$$S_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi(\frac{U-\mu}{\sigma}) + \frac{1}{2}\Phi(\frac{\mu-L}{\sigma})\right\} \tag{24}$$

such that, for $S_{pk} = c$ and the PNC will be $2\{1-\Phi(3c)\}$.

Chen, Pearn, and Lin (2003) defined a multivariate analogue of S_{pk}, for the situation where all the quality characteristics are mutually independent as,

$$S_{pk}^T = \frac{1}{3}\Phi^{-1}\left\{1 + \left[\prod_{j=1}^{q}(2\Phi(3S_{pkj}) - 1)\right]/2\right\}, \tag{25}$$

where S_{pkj} is the value of S_{pk} for the jth quality Characteristic, for $j = 1(1)q$.

Since the quality characteristics of a multivariate process are seldom mutually independent, Pearn, Wang, and Yen (2006) have considered the concept of principal component analysis to define the following MPCI, which does not require the independence assumption

$$TS_{pk,pc} = \frac{1}{3}\Phi^{-1}\left\{\left[\prod_{j=1}^{q}\left(2\Phi(3S_{pk_j,pc}) - 1\right) + 1\right]/2\right\} \tag{26}$$

where $S_{pk_j,pc}$ denotes the S_{pk} value for the jth PC, $j = 1(1)q$.

However note that, ideally in Eq. (26), q should be replaced by "v," the number of significant PCs.

The authors have also shown that similar to the case of S_{pk}, for $TS_{pk,pc}$, $PNC = 2\left\{1 - \Phi(3TS_{pk,PC})\right\}$.

Interestingly, although application of PCA does not require multivariate normality assumption, since S_{pk} is defined for normally distributed quality characteristics, in general, $TS_{pk,pc}$ requires the multivariate normality assumption of the process distribution to be fulfilled.

After applying $TS_{pk,pc}$ to the transformed data given in Table 2, it is found that $TS_{pk,pc} = 1.0632$ and $PNC = 0.0014$ which indicates toward the satisfactory performance of the process and hence $TS_{pk,pc}$ clearly overestimates process capability.

3.3.2. Gonzalez and Sanchez's (2009) MPCI

Gonzalez and Sanchez (2009) have proposed univariate and multivariate PCIs which are directly related to the concept of PNC. For this purpose the authors have considered PCIs which are based only on process variance. For a process with "q" correlated quality characteristics, let the tolerance region be defined as

$$S = \{\boldsymbol{X} \in R^q : (LSL_i \leq X_i \leq USL_i), i = 1(1)q\}.$$

The authors defined $\Sigma_{\max}$ as the maximum allowable dispersion, i.e., the dispersion related to $\boldsymbol{X}_{\max}$, where $\boldsymbol{X}_{\max}$ has density similar to $\boldsymbol{X}$ with $P(\boldsymbol{X}_{\max} \notin S) = \alpha$.

For a multivariate normal process, the authors have used PCA to obtain PCs which can be considered as the primary independent factors contributing to the variability of the process. The authors considered the following two situations:

1. Variability of only one factor changes at a time.
2. Variability of all the factors change simultaneously.

For the first case, let the change in the variability of a single factor, say the ith factor be incorporated through multiplying the corresponding eigen value λ_i by a factor b_i^2, $i = 1(1)q$. Consequently, a new matrix can be defined as $D^*_{(i)} = diag(\lambda_1, \lambda_2, \ldots \lambda_{i-1}, b_i^2\lambda_i, \lambda_{i+1}, \ldots, \lambda_q)$ such that $\boldsymbol{X}^*_{(i)} \sim N_q(\boldsymbol{\mu}, \Sigma^*_{(i)})$ with $\Sigma^*_{(i)} = CD^*_{(i)}C'$, where C is the $(q \times q)$ matrix of eigen vectors of Σ. Let $P^*_{(i)} = P(\boldsymbol{X}^*_{(i)} \ni S)$ and define $b_i^{\max}$ such that $P^*_{(i)} = \alpha$.

Then Gonzalez and Sanchez (2009) have defined an MPCI, namely $C_{n,i} = \left\{\frac{|\Sigma^{\max}_{(i)}|}{|\Sigma|}\right\}^{1/2} = b^{\max}_{(i)}$, where $\Sigma^{\max}_{(i)}$ is defined similar to $\Sigma_{(i)}$, the only difference being here $b_{(i)}$ is replaced by $\ni^{\max}_{(i)}$.

The authors have considered the process as capable if $C_{n,i} \geq 1$ for $i = 1(1)q$. Therefore, for an incapable process, the responsible factor(s) can be readily identified by merely observing the values of $C_{n,i}$.

For the next case, i.e., for the case of simultaneous change in the values of all the variables, the authors have assumed same amount of change in all the variables and have defined $C_n^S = \left\{\frac{|\Sigma^{\max}|}{|\Sigma|}\right\}^{\frac{1}{2v}} = b^{\max}$, where $b^{\max}$ is defined such that $P(\boldsymbol{X}^{\max} \in S) = \alpha$.

For the multivariate non-normal processes, Gonzalez and Sanchez (2009) have advocated the use of an independent component analysis (ICA) as according to them PCA is not applicable to multivariate non-normal processes. However, following Johnson and Wicharn (2013), PCA is applicable to multivariate non-normal processes as well.

In the context of Sultan's transformed data given in Table 2, $C_{n,1} = b_1^{\max} = 0.0745 < 1$ which indicates toward the incapability of the process. Also the maximum value of b_1 is 0.0745 which is quite small.

Since using $C_{n,1}$, the process is found to be incapable, hence there is no point in computing $C_{n,2}$.

3.3.3. Pan and Chen's (2012) concept of expected loss of quality

Pan and Chen (2012) have established relationships between expected loss in quality and some popular MPCIs, namely MC_{pm} [vide equation (15)] and NMC_{pm} [vide Eq. (19)]. In this context, both MC_{pm} and NMC_{pm} are defined as multivariate analogue of C_{pm}, which in turn, deals with expected loss in quality due to deviation from 'T' in case of univariate processes.

Suppose C is a $q \times q$ p.d. matrix of costs, which represents losses incurred due to deviation of $\boldsymbol{X}$ from $\boldsymbol{T}$.According to Pan and Chen (2012), if "C" is a diagonal matrix, then the corresponding loss function is the sum of "q" single response quadratic loss functions. On the other hand, if "C" is not a diagonal matrix, then the off-diagonal elements represent the incremental losses incurred when the ith and jth pair of quality characteristics are simultaneously off-target.

In this context, following Pignatiello (1993) multivariate quality loss function can be defined as $Q = (\boldsymbol{X} - \boldsymbol{T})'C(\boldsymbol{X} - \boldsymbol{T})$ with

$$E(Q) = (\boldsymbol{\mu} - \boldsymbol{T})'C(\boldsymbol{\mu} - \boldsymbol{T}) + tr(C\Sigma). \tag{27}$$

Note that, for $\boldsymbol{\mu} = \boldsymbol{T}$, $E(Q) = tr(C\Sigma)$.

Pan and Chen (2012) have established a relationship between the expected multivariate quality loss function $E(Q)$ and MC_p as

$$E(Q) = tr(C\Sigma) = \frac{|C|\,|M|}{MC_p^2} \times \sum_{j=1}^{q} \frac{1}{|\Lambda_{(j)}|}, \tag{28}$$

where, $M = \text{diag}\left(\frac{d_i^2}{\chi^2_{q,0.9973}}\right)$, $\chi^2_{q,0.9973}$ is the 99.73^{th} percentile of χ^2 distribution with "q" degrees of freedom, $d_i = \frac{USL_i - LSL_i}{2}$ for $i = 1(1)q$, $|\Lambda_{(j)}| = \prod_{k=1;k\neq j}^{q} \lambda_k$ and λ_j is the jth eigen value of $C\Sigma$, for $j = 1(1)q$.

Similarly, for nominal the best-type quality characteristics the authors have shown that

$$E(Q) = k_N \left[\left(\frac{MC_p}{MC_{pm}} \right)^2 - 1 \right] + \frac{1}{MC_p^2} \left(|C|\, |M| \sum_{j=1}^{q} \frac{1}{|\Lambda_{(j)}|} \right) \tag{29}$$

where $k_N = \frac{(\boldsymbol{\mu}-\boldsymbol{T})' C (\boldsymbol{\mu}-\boldsymbol{T})}{(\boldsymbol{\mu}-\boldsymbol{T})' \Sigma^{-1} (\boldsymbol{\mu}-\boldsymbol{T})}$.

Finally, for NMC_p and NMC_{pm}, defined in Eqs. (18) and (19), respectively, the authors have shown that

$$E(Q) = k_N \left[\left(\frac{NMC_p}{NMC_{pm}} \right)^2 - 1 \right] + \frac{1}{NMC_p^2} \left(|C|\, |A^*| \sum_{j=1}^{q} \frac{1}{|\Lambda_{(j)}|} \right) \tag{30}$$

While validating their proposed theory. Pan and Chen (2012) have considered Sultan's (1986) original dataset (assuming multivariate normality of the underlying process distribution) and have considered the cost matrix as $C = \begin{pmatrix} 0.8 & 0.89 \\ 0.89 & 1 \end{pmatrix}$ with $|C| = 0.0079$.

While applying their theory to the transformed data given in Table 2 we have retained this "C" matrix. Here, $|A^*| = 9721538$ and $K_N = 17617.79$. We have already computed $\hat{N}MC_p = 1.1156$ and $\hat{N}MC_{pm} = 1.1035$. Hence, $E(Q) = 91835.42$ which suggests that the expected loss of quality is quite high, which contradicts the high values of NMC_p and NMC_{pm} as both of them are greater than 1. Thus, NMC_p and NMC_{pm} tend to over-estimate process capability.

Note that Pan and Chen (2012) have computed the value of $E(Q)$ as 462.256 which also seems to be quite high. However, due to unavailability of any threshold value of $E(Q)$ in their proposed formulations, it is difficult to draw any concrete conclusion about the degree of quality loss, though apparently, the process is not maintaining a good health.

3.4. Vector-valued MPCIs

Although process capability indices (both univariate and multivariate) are generally considered as a single-valued assessment of process performance, sometimes it is difficult to assess the overall capability of a multivariate process through such conventional single-valued indices. Therefore, a number of vector valued MPCIs have been defined in the literature to deliver the good (refer to Kotz and Johnson 1992-2000; Pearn and Kotz 2007 and the references there in). We shall now revisit a few of those multivariate process capability vectors (MPCV) in the light of Sultan's (1986) data.

3.4.1. Shahriari, Hubele, and Lawrence's (1995) MPCV

Shahriari, Hubele, and Lawrence (1995) were probably the first to introduce the concept of multivariate process capability vector (MPCV) in the literature. Their proposed MPCV consists of three components.

The first component, called C_{pM}, is analogous to C_p of univariate processes. Suppose a process has "q" correlated quality characteristics and "n" items are randomly selected from the process. For each of these "n" items, measurements are taken for all the "q" quality characteristics. Then, C_{pM} can be defined as

$$C_{pM} = \left[\frac{\text{Volume of engineering tolerance}}{\text{Volume of modified process region}}\right]^{\frac{1}{q}} = \left[\frac{\prod_{i=1}^{q}(USL_i - LSL_i)}{\prod_{i=1}^{q}(UPL_i - LPL_i)}\right]^{\frac{1}{q}}, \tag{31}$$

where $UPL_i = \mu_i + \sqrt{\frac{\chi^2_{q,\alpha}|\Sigma_i^{-1}|}{|\Sigma^{-1}|}}$ and $LPL_i = \mu_i - \sqrt{\frac{\chi^2_{q,\alpha}|\Sigma_i^{-1}|}{|\Sigma^{-1}|}}$,for $i = 1(1)q$; $\chi^2 q, \alpha$ is the upper α quantile of a χ^2 distribution with "q" degrees of freedom associated with the probability contour and Σ_i^{-1} is obtained by deleting the ith row and the ith column from Σ^{-1}, for $i = 1(1)q$. The observed value of C_{pM} greater than 1 indicates that the circumscribed modified process region is smaller than the engineering tolerance region and hence the process is performing satisfactorily.

Assuming that the center of the engineering specification region is the true mean of a process ($\boldsymbol{\mu}_0$), the second component is defined as

$$PV = P\left[T^2 > \frac{q(n-1)}{n-q} F_{q,n-q}\right], \tag{32}$$
$$\text{where } T^2 = n(\overline{\mathbf{X}} - \boldsymbol{\mu}_0)' S^{-1}(\overline{\mathbf{X}} - \boldsymbol{\mu}_0),$$

and 'S' is the sample variance–covariance matrix.

Finally, the third component of the MPCV is

$$LI = \begin{cases} 1, & \text{if the modified process region is entirely within the tolerance region,} \\ 0, & \text{Otherwise.} \end{cases}$$

For the data in Table 2, $C_{pM} = 1.0562$ which indicates toward satisfactory process performance. Also for these data. PV = 0.7703. Since according to Wang, Hubele, and Lawrence (2000), value of PV close to 1 indicates that the process center is close to **T**, it is logical to expect that the process center is fairly close to **T**. However, a threshold value needs to be defined for PV, beyond which the process centering can be considered to be close to $\boldsymbol{T}$. This, in turn, will reduce subjectivity of the index.

As far as the third component LI is concerned, Pan and Lee (2010), while dealing with Sultan's (1986) original dataset, considered $LI = 1$ as for that data set, modified process region is completely within the tolerance region.

However, for the transformed data, which we are presently dealing with, the situation is some what different. For brinell hardness, $(LPL_1, UPL_1) = (112.9686, 239.4314)$ is within $(USL_1, LSL_1) = (111.7, 240.3)$. But for tensile strength, $(LPL_2, UPL_2) = (403.3929, 2364.851)$ while $(LSL_2, USL_2) = (534.145, 2685.945)$. Hence, the modified process region is not completely within the tolerance region and therefore by definition, $LI = 0$. Therefore, the final form of the MPCV is $(1.0562, 0.7703, 0)$.

Thus, Shahriari, Hubele, and Lawrence's (1995) MPCV is somewhat self contradictory in a sense that, while the first two components apparently give an impression of satisfactory process performance, the third component opines differently. Hence in general, the process performance should not be considered as satisfactory.

3.4.2. Shahriari and Abdollahzadeh's (2009) MPCV

Shahriari and Abdollahzadeh (2009) have taken into account the concepts of two major MPCIs defined by Taam, Subbaiah, and Liddy (1993) and Shahriari, Hubele, and Lawrence (1995).

Modifying the first component MC_{pM} of Shahriari, Hubele, and Lawrence's (1995) MPCV, similar to that of Taam, Subbaiah, and Liddy's (1993) MC_{pm}, they have defined

$$NMC_{pM} = \frac{C'}{\sqrt{\chi^2_{q,0.0027}}} \times D, \tag{33}$$

where $D = \left[1 + (\boldsymbol{\mu} - \boldsymbol{T})'\Sigma^{-1}(\boldsymbol{\mu} - \boldsymbol{T})\right]^{-\frac{1}{2}}$. Also, $C' = \min_i \left\{C_i = \frac{U_i - T_i}{\sigma_i}, i = 1(1)q\right\}$ for symmetric specifications and $C' = \min\left[\min_i \left\{C_i = \frac{U_i - T_i}{\sigma_i}, \frac{T_i - L_i}{\sigma_i}\right\}, i = 1(1)q\right]$ for asymmetric specification limits.

The authors retained the other two components of Shahriari, Hubele, and Lawrence's (1995) MPCV, viz, PV and LI.

After transformation into multivariate normality, the transformed specification region becomes asymmetric with respect to the transformed target vector. Hence following Shahriari and Abdollahzadeh (2009), $C' = 3.0505$ and hence $NMC_{pM} = 0.8869$. Since Shahriari and Abdollahzadeh (2009) have retained the other two components, the computed value of their MPCV for the data in Table 2 is [0.8869, 0.7703, 0] and this clearly suggests that the process is not performing satisfactorily.

Thus, Shahriari and Abdollahzadeh's (2009) MPCV out performs Shahriari, Hubele, and Lawrence's (1995) MPCV in a sense that in case of the former, there is no case of contradiction between the components of the MPCV.

Interestingly, the second component in both the MPCVs indicates that the process centering is fairly close to the target vector, while for Shahriari and Abdollahzadeh's (2009) MPCV, the first and the third components suggest that the modified process region is not completely within the specification region. Combining these two findings one can conclude that it is the process dispersion which requires immediate attention of the concerned authority for smooth running of the process.

3.4.3. Ciupke's (2015) MPCV

Ciupke (2015) has proposed an MPCV having three components, which play roles similar to those of the MPCV proposed by Shahriari, Hubele, and Lawrence (1995). In order to accommodate both normal and non-normal multivariate distributions and the correlated and uncorrelated data, Ciupke (2015) has considered "one-sided model". In this context, one-sided models address the so-called problem of unknown model accuracy in case of fitting a regression model.

The MPCV proposed by Ciupke (2015) consists of three components viz. (C_{PV}, PS and PD). Here, C_{PV} is defined as the ratio of volume of the tolerance region (R_T) to the volume of the process region (R_P). Thus, $CPV = \left\{\frac{VOL.(R_T)}{VOL.(R_P)}\right\}^{1/q} \times 100\%$ with $VOL.(R_T) = \prod_{i=1}^{q}(USL_i - LSL_i)$.

Suppose there are two variables X_1 and X_2 in the regression model with X_1 being the independent variable. Then in one-sided model, to address the problem of model accuracy, a pair of functions $\{\hat{x}2^-(x_1), \hat{x}2^+(x_1)\}$ (or simply $\{\hat{x}2^-, \hat{x}2^+\}$) is used to limit data from the bottom ($\hat{x}2^-$) and from the top ($\hat{x}2^+$).

To compute the volume of the process region, Ciupke (2015) first considered an independent variable, say X_k, on the basis of the knowledge of the process and then based on X_k, setup a pair of one-sided models $\hat{\boldsymbol{X}} = \bigcup_{i \in I}\{\hat{X}_i^-, \hat{X}_i^+\}$, where $I = \{1, 2, \cdots, q\}$ such that $k \ni I$.

For bivariate processes, $VOL.(R_P)$ can be computed analytically. However, for processes with higher dimension of Ciupke (2015) has suggested the use of numerical integration and has provided an approximate expression for $VOL.(R_P)$ in this regard. The value of $VOL.(R_P)$ is smaller the better with $CPV = 100\%$ indicating that the process region coincides with the tolerance region.

The second component of the MPCV, namely PS measures the shift of process median with respect to the target and can be mathematically formulated as

$$PS = \max_{i=1(1)q}\left(\frac{|T_i - m_i|}{D_i}\right) \times 100\%, \tag{34}$$

where $m_i = median(X_i)$, $i = 1(1)q$ and $D_i = \begin{cases} U_i - T_i, & \text{if } m_i \geq T_i,\ i = 1(1)q \\ T_i - L_i, & \text{Otherwise.} \end{cases}$.

For a process perfectly centered at $\mathbf{T}$, $PS = 0\%$ and on the contrary, if the process median coincides with one of the specification limits, then $PS = 100\%$.

In this context, usually, for multidimensional processes, the underlying process distribution is assumed to be multivariate normal to retain computational simplicity and hence shift of the process mean vector from $\mathbf{T}$ is considered. However, since Ciupke (2015)'s MPCV accommodates processes with multivariate non-normal distributions as well, median is considered instead of mean. Note that, for multivariate normal process, for each individual quality characteristic, mean coincides with median.

The third component of Ciupke's (2015) MPCV is 'PD' which is defined to capture process dispersion, as two processes may have the same values of CPV and PS, while their dispersion scenario may be completely different all together. Ciupke (2015) has defined PD as

$$PD = \max_{i=1(1)q}\left[\frac{\max(\hat{\boldsymbol{X}}_i^{+}) - \boldsymbol{T}}{\boldsymbol{U} - \boldsymbol{T}}, \frac{\boldsymbol{T} - \max(\hat{\boldsymbol{X}}_i^{-})}{\boldsymbol{T} - \boldsymbol{L}}\right] \times 100\% \tag{35}$$

Thus, PD $<$ 100% implies that the process region is within the specification region while PD $>$ 100% indicates that the process region is not completely included in the specification region. However, Ciupke (2015) has opined that for practical purpose, PD value close to 100%, though less than 100% may be considered as indication towards possible incapability of the process.

Since the MPCV suggested by Ciupke (2015) does not consider any distributional assumption, it rightly judges the data of Sultan (1986) to correspond to an incapable process as PD value is quite high (96%) indicating toward the high degree of process variability.

3.4.4. Polansky's (2001) MPCI

Polansky (2001) has adopted nonparametric approach, to be more precise kernel estimation approach, to define a proportion of nonconformance (PNC)-based MPCI.

For a "q" variate random vector $\mathbf{X}$, characterizing multiple quality characteristics with the corresponding pdf $f(\boldsymbol{x}) = f(x_1, x_2, \cdots, x_q)$ and cdf $F(\boldsymbol{x}) = F(x_1, x_2, \cdots, x_q)$ and for a specification set $S \subset R^q$, PNC can be defined in general as

$$p = P(\boldsymbol{X} \ni S) = 1 - \int_S f(\boldsymbol{x})d\boldsymbol{x} \tag{36}$$

Since "p" is often unknown in practice, Polansky (2001) has adopted nonparametric kernel estimation procedure to estimate the same.

Let "K" be a continuous "q" variate kernel function satisfying the following conditions:

1. $\int_{R^q} K(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 1;$

2. $\int_{R^q} \boldsymbol{x}K(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0$;
3. $\int_{R^q} \boldsymbol{x}\boldsymbol{x}'K(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \mu_2(K)I_q$,

where $\mu_2(K) = \int_{R^q} x_i^2 K(\boldsymbol{x})d\boldsymbol{x}$ assumes the same value for all 'i' and I_q is an $(q \times q)$ identity matrix. Also, for an $(q \times q)$ symmetric positive-definite matrix H, which is called the band width matrix, let us define

$$K_H(\boldsymbol{x}) = |H|^{-1/2} \times K(H^{-1/2}\boldsymbol{x}) \tag{37}$$

and the q-variate kernel density estimator of 'f' as

$$\hat{f}(\boldsymbol{x}, H) = n^{-1} \sum_{i=1}^{n} K_H(\boldsymbol{x} - \boldsymbol{X}_i), \forall \boldsymbol{x} \in R^q. \tag{38}$$

Then Polansky (2001) has shown that

$$\hat{p}(H, S) = 1 - \int_S \hat{f}(\boldsymbol{x}; \boldsymbol{H})d\boldsymbol{x} = 1 - n^{-1} \sum_{i=1}^{n} \int_S K_H(\boldsymbol{x} - \boldsymbol{X}_i)d\boldsymbol{x}. \tag{39}$$

He has also computed the expression for MSE of $\hat{p}(H, S)$ and its estimator using a smoothed least-square crossvalidation technique.

Finally, Polansky (2001) defined a nonparametric MPCI, similar to C_p as $\hat{C}_p^M = -\frac{1}{3}\Phi^{-1}[\hat{p}(H, S)]$, although the interpretation of $\hat{C}_p^M$ is not identical to that of C_p.

Interestingly, for Sultan's (1986) data, Polansky (2001) computed $\hat{p}(H, S) = 0.00758$ and $\hat{C}_p^M = 0.81$ which shows that the process is not performing satisfactorily. In this context, recall that Polansky (2001) addressed the problem from nonparametric view point and hence did not make any distributional assumption while assessing capability of the process.

According to Polansky (2001), the presence of a point near the boundary of the rectangular specification region indicates that a significant amount of tail area of the process distribution may overlap the boundary leading toward poor process capability value. This is actually the case here, as can be seen in Figures 1 and 2.

However, the hybrid bootstrap confidence interval for $\hat{p}(H, S)$ is $[0, 0.1372]$, which is fairly wide and according to Polansky(2001), this happens due to small sample size.

3.5. A consolidated discussion

Following is a brief discussion regarding performance of the MPCIs discussed so far, from the perspective of the impact of distributional assumptions on the MPCIs and Sultan's (1986) transformed data [vide Table 2].

1. MPCIs proposed by Wang and Chen (1998) over estimate process capability, even after the data are transformed into multivariate normality.
2. Both the MPCIs given by Tano and Vannman (2013) and Wang and Du (2000) rightly conclude that the process is incapable.

   However, Wang and Du (2000)'s MPCIs are somewhat easy to apply, since unlike those given by Tano and Vannman (2013), their MPCIs do not require the so-called standardization of the data and can be applied even if the data is non-normal.

   Interestingly, for the transformed and standardized data, only 92.35 % of the total variability is explained by the first principal component, while the remaining 7.65 % of the total variability still remain unexplained.

   If the original data (in Table 1) are standardized, as is done by Tano and Vannman (2013), the explained variation by the first principal component will be only 91.69 %, i.e., the situation will be even worse.

   Also, for Sultans's (1986) data, after transformation to normality, its specification becomes asymmetric. Tano and Vannman's (2013) MPCI does not take this into consideration. Our proposed modification in this regard, viz. $C''_{p,TV}$ solves this issue.
3. For Taam, Subbaiah, and Liddy's (1993) MC_{pm}, modified tolerance region is independent of the correlation structure prevailing in the process and consequently MC_{pm} ends up with over estimating process capability, particularly, when two or more of the quality characteristics are highly correlated (refer to Pan and Lee 2010; Shahriari and Abdollahzadeh 2009). Therefore, for Sultan's (1986) transformed data, MC_{pm} fails to perform satisfactorily.

   To address this problem, Pan and Lee (2010) defined NMC_p and NMC_{pm}. Although these MPCIs perform better than MC_p and MC_{pm}, respectively, they still over-estimate process capability.
4. According to Goethal and Cho (2011), the process is found to be incapable (due to high variability), though the corresponding MPCI value is still very high ($MC_{pmc} = 5.015$). Our suggested modification scales down the value ($MC^*_{pmc} = 1.6249$) to some extent.
5. Chen's (1994) MPCI, namely MC_p^{chen} considers the process to be capable both before and after transformation of Sultan's (1986) data and hence it is logical to conclude that MC_p^{chen} over-estimates process capability.
6. The super-structure of MPCIs viz. $C_M(u,v)$, for $u = 0,1$ and $v = 0,1$ proposed by Chatterjee and Chakraborty (2017) considers the process to be incapable. Availability of the threshold value, in this regard, makes the decision unambiguous.
7. Among the MPCIs based on the concept of proportion of nonconformance, Pearn, Wang, and Yen's (2006) MPCI over-estimates process capability, while the MPCI defined by Gonzalez and Sanchez (2009) rightly indicates towards poor capability of the process.

For the transformed data, the expected loss of quality, computed following Pan and Chen (2012), is quite high—which means the process suffered from poor quality. But due to unavailability of any threshold value of $E(Q)$, it is difficult to make any precise comment.

8. Both the MPCVs defined by Shahriari, Hubele, and Lawrence (1995) and Shahriari and Abdollahzadeh (2009) consider the process to be performing unsatisfactorily.
9. The MPCIs defined by Ciupke (2015) and Polansky (2001) do not require any distributional assumption and they both considered the process to be of a poor capability level.
10. Unavailability of mathematical expression for the threshold values of MC_{pm}, MC_{pmc} and $E(Q)$—the expected quality loss defined by Pan and Chen (2012), make it difficult to take clear-cut decision regarding the actual capability of the process.

The above discussion is summarized in Table 3 to enable easy comparison between MPCIs discussed in this article.

4. Concluding remark

Sultan's (1986) data have been used time and again in the literature with multivariate normality of the underlying process. However, as has been discussed in detail in Sec. 2, this is not a valid assumption. Polansky has shown that the performance analysis of the process, which Sultan's (1986) data represent, differs a significantly under multivariate normality assumption and under nonparametric setup. Tano and Vannman (2013) and Shahriari and Abdollahzadeh (2009) too were skeptical about the validity of multivariate normality assumption. However, Tano and Vannman (2013) restricted themselves to checking univariate normality of the individual quality characteristics, viz. brinell hardness and tensile strength, instead of carrying out the desired multivariate normality test.

Based on the discussion in this article, following recommendations can be made regarding choice of MPCI:

i) MPCIs based on nonparametric approaches, namely Ciupke's (2015) multivariate process capability vector and Polansky's (2001) kernel-based MPCI, do not require any distributional assumption and hence are safer to use. However, these are computation-centric and therefore, the user should be comfortable with various methods of statistical computing.
ii) While using MPCIs based on parametric approaches, one must ascertain the underlying process distribution before its actual use. To be more precise, Wang and Du (2000) and Gonzalez and Sanchez (2009) defined MPCIs for multivariate normal and non-normal distributions. When applied to

Table 3. Comparison of the performance of MPCIs discussed in this article

Name of MPCI and corresponding author(s)	Whether multivariate normality is checked initially	Conclusion Drawn by respective author(s) regarding process performance	Our observation(s) after transformation of data to multivariate normality (If required)
MC_p, MC_{pk}, MC_{pm}, MC_{pmk} [Wang and Chen (1998)]	No	Capable: MPCI values are almost equal, implying Process mean is close to target	The index Over-estimates process capability (even after transforming data to multivariate normality)
$C_{p,TV}$ [Tano and Vannman (2013)]	No	Not Capable	Our proposed modification, namely $C''_{p,TV}$, will be applicable here, as the transformation makes the specification limits asymmetric with respect to target
MC_{pc} [Wang and Du (2000)]	No	—- [Wang and Du (2000) did not use MC_{PC} as they assumed the data to be multivariate normal]	The data, originally, being non-normal, MC_{PC} performs well on this
MC_p & MC_{pm} [Taam, Subbaiah, and Liddy (1993)]	No	Capable	The index Over-estimates process capability (even after transforming data to multivariate normality)
NMC_p & NMC_{pm} [Pan and Lee (2010)]	No	Capable	The index Over-estimates process capability (even after transforming data to multivariate normality)
MC_{pmc} [Goethals and Cho (2011)]	No	Not Capable Highly off-centered and high variability	Our proposed scaled-down version MC^*_{pmc} performs better
MC_p [Chen (1994)]	No	Capable	The index Over-estimates process capability (even after transforming data to multivariate normality)
$C_M(u,v)$ [Chatterjee and Chakraborty (2014)]	Yes	Not capable highly off centred and high variability	The index performs well for multivariate normal data
$TS_{pk;pc}$ [Pearn, Wang, and Yen (2006)]	No	Capable	The index Over-estimates process capability (even after transforming data to multivariate normality)
$C_{n,i}$ [Gonzalez and Sanchez (2009)]	No [However can accommodate multivariate non-normal data as well]	Not capable [Although this conclusion was based on multivariate normality of the data]	The index performs well (even when the data are transformed to normality)
$E(Q)$ i.e. expected quality loss [Pan and Chen (2012)]	No	High quality loss	The index performs well for the transformed data also; but requires additional information on cost of production
$[C_{pM}, PV, LI]$ [Shahriari, Hubele, and Lawrence (1995)]	No	—- [Shahriari, Hubele, and Lawrence (1995) did not analyze the data themselves]	First and third component values are self-contradictory

(Continued)

Table 3. Continued

Name of MPCI and corresponding author(s)	Whether multivariate normality is checked initially	Conclusion drawn by respective author(s) regarding process performance	Our observation(s) after transformation of data to multivariate normality (if required)
$[NC_{pM}, PV, LI]$ [Shahriari and Abdollahzadeh (2009)]	No	Not capable	The index performs well after transforming the data to multivariate normality also
[CPV, PS, PD] Ciupke (2015)	Does not require any distributional assumption	Not Capable	The process capability vector performs satisfactorily
$\hat{p}(H, S)$ Polansky (2001)	Does not require any distributional assumption	Not Capable	The process capability vector performs satisfactorily

Sultan's (1986) original multivariate non-normal data, their MPCIs for non-normal distributions are found to perform satisfactorily as well. However, since Wang and Du (2000) and Gonzalez and Sanchez (2009) themselves did not carry out multivariate normality test and considered the data to be multivariate normal, they failed to get accurate result.

iii) Among the other parametric MPCIs, those proposed by Chatterjee and Chakraborty (2014), Shahriari and Abdollahzadeh (2009), and Pan and Chen (2012) are found to perform satisfactorily for multivariate normal data. MPCIs given by Tano and Vannman (2013), Goethals and Cho (2011) also perform well and our proposed modifications over those MPCIs are seen to make them perform better.

iv) Other MPCIs, discussed in this article, are found to have tendency of over-estimating process capability and hence should be used cautiously.

As has been discussed in this article, incorrect assumption regarding the distribution of a process may lead to erroneous decision about the capability of that process. Therefore, it is utmost necessary to be certain about the underlying distribution of the process, which the data are representing and select the MPCI accordingly. The study also revealed the tendency of some MPCIs to over-estimate process capability. In a nut-shell, one should not decide about performance of a process merely from an MPCI value—proper exploration of all the prospective dimensions of the problem is utmost solicited.

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