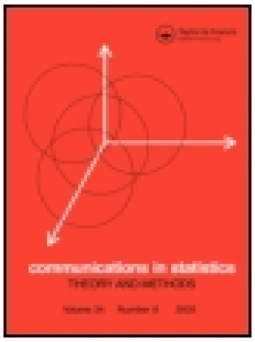

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Some Process Capability Indices for Unilateral Specification Limits - Their Properties and the Process Capability Control Charts

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Some Process Capability Indices for Unilateral Specification Limits - Their Properties and the Process Capability Control Charts

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Short Title: *Distributions and Capability Control Charts for* $C_p^U(u,v)$ *and* $C_P^L(u,v)$

Abstract

Processes with smaller the better and larger the better types of quality characteristics and consequently the unilateral specification limits are very common in manufacturing industries. However, very little theoretical resources are available in literature, compared to the bilateral specification limits, for assessing the capability of such processes. In the present article, we have studied the expressions for the threshold value and relationship with proportion of non-conformance for some of the process capability indices (PCI) for unilateral specification limits. We have also explored the distributional aspects along with the uniformly minimum variance unbiased estimators of those PCIs based on both single sample information as well as the information obtained from the corresponding $\overline{X}-R$ and $\overline{X}-S$ control charts. The process capability control charts for these PCIs have been designed as well for the purpose of the continuous assessment of the capability of a process over the entire production cycle. Finally, a numerical example has been discussed in the context of the theory developed in this article.

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1 Introduction

Process Capability Indices (PCI) assess the ability of a process to produce items within pre-assigned specification limits, viz., upper specification limit (USL) and lower specification limit (LSL). With the increasing emphasis on quality in world economy, application of various PCIs for correct assessment of the capability of different processes generated from several diversified fields, especially manufacturing industries, is increasing day by day.

From the view point of the nature of specification limits, most of the quality characteristics can be broadly classified into either of the following three classes:

1. The nominal the best (Processes with both USL and LSL), e.g. height, length;

2. The smaller the better (Processes with only USL), e.g. surface roughness, flatness;

3. The larger the better (Processes with only LSL), e.g. tensile strength, compressive strength.

Under the assumption of normality of the concerned quality characteristic (X), the four classical PCIs for processes with bi-lateral specification limits are $C_p = \frac{U-L}{6\sigma}, C_{pk} = \frac{d-|\mu-M|}{3\sigma}, C_{pm} = \frac{d}{3\sqrt{\sigma^2+(\mu-T)^2}}$ and $C_{pmk} = \frac{d-|\mu-M|}{3\sqrt{\sigma^2+(\mu-T)^2}}$. Here, 'U' and 'L' denote the USL and LSL respectively; $d = (U-L)/2, M = (U+L)/2$, 'T' denotes the targeted value of the quality characteristic under consideration and μ and σ denote the mean and variance of the quality characteristic, such that, $X \sim N(\mu, \sigma^2)$.

Although most of the PCIs defined so far are meant for nominal the best type of quality characteristics [see Kotz and Johnson (2002) and the references there-in], some very useful research works are also available in literature for processes with unilateral specification limits. Chatterjee and Chakraborty (2012) have made an extensive review of these PCIs.

The research works on PCIs carried out in the field of unilateral specification limits are mostly based on the two indices $C_{PU} = \frac{U-\mu}{3\sigma}$ and $C_{PL} = \frac{\mu-L}{3\sigma}$ due to their computational simplicity. However, neither of C_{PU} or C_{PL} incorporate the concept of target (T) in their definitions. Moreover, unlike

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C_p, C_{PU} and C_{PL} do not even measure the potential capability of a process as these PCIs are expressed as functions of the process centering (μ). Note that, by the term 'potential capability' we mean the capability level, that a process can at most attain given the current dispersion level and specification scenario.

In this context, the concept of 'T', though not always explicitly discussed in the cases the of larger the better and smaller the better types of quality characteristics, it actually has huge impact on these types of quality characteristics. For example, let us consider purity of gold as the quality characteristic of concern. This is a larger the better type of quality characteristic. For making ornaments, a purity of 91.66% – 95.83% (i.e. 22 carat) is sufficient. Moreover for gold with higher degree of purity the cost of production is exorbitant and it finds very limited application - mostly in cutting-edge laboratories for the purpose of sophisticated experimentations of physical and chemical sciences. Hence, a jeweler should target at producing gold with 91.66% – 95.83% purity because even if the purity of his gold is higher than the targeted value, the customer will not be ready to spend extra money on that. Also, gold of higher purity, say 24 carat, tend to be more fragile than 22 carat gold and hence is not suitable for making ornaments.

We can also consider the example of surface roughness, in this regard, which is a quality characteristic of smaller the better type. Theoretically, surface roughness should be as small as possible, though in practice for a surface with the roughness below a certain level will not only increase the cost of production, but the surface will also become slippery beyond manageability and hence may not be useful in day to day activities.

Hence, proper setting of target is a key to successful operation of a process. Unfortunately, C_{PU} and C_{PL}, the most primitive and widely used PCIs for unilateral specifications do not incorporate the concept of 'T' in their definitions.

To address the problems of C_{PU} and C_{PL}, Vännman (1998) suggested two sets of superstructures of PCIs for unilateral specification limits. However, these super-structures suffer from a number of major drawbacks such as, imposing equal amount of importance on deviation of μ towards either side of 'T'; inability to have maximum index value on target, obtaining negative value of the index before μ reaches U or L and so on [see Grau (2009) for more details]. Grau (2009) defined the following superstructures of PCIs which are free from these drawbacks:

$$\left.\begin{aligned} C_p^U(u,v) &= \frac{U-T-uA_U^*}{3\sqrt{\sigma^2+vA_U^{*2}}} \\ \\ C_p^L(u,v) &= \frac{T-L-uA_L^*}{3\sqrt{\sigma^2+vA_L^{*2}}} \end{aligned}\right\} \quad (1)$$

where, $A_U^* = \max\{(\mu - T), \frac{T-\mu}{k}\}$, $A_L^* = \max\{\frac{\mu-T}{k}, (T-\mu)\}$ and 'u' and 'v' are two non-negative parameters. Also, the value of $k\ (>1)$ quantifies the risk of deviation from the target in the direction opposite to the available specification limit with respect to 'T'. Note that $C_p^U(0,0) = C_p^U, C_p^U(1,0) = C_{pk}^U, C_p^U(0,1) = C_{pm}^U$ and $C_p^U(1,1) = C_{pmk}^U$ are defined analogous to C_p, C_{pk}, C_{pm} and C_{pmk}. Similar is the case for $C_p^L(u,v)$. To avoid notational ambiguity, let us define $C_p^I(u,v)$ which stands for $C_p^U(u,v)$ and $C_p^L(u,v)$ depending on the available specification limit. Here the additional tuning parameters viz., u and v are introduced to unify the PCIs C_p^I, C_{pk}^I, C_{pm}^I and C_{pmk}^I for ease of representation.

Note that, $C_p^I(u,v)$ is not a new class of PCIs for unilateral specification limits. It only works as an indicator of whether $C_p^U(u,b)$ or $C_p^L(u,v)$ will be appropriate to use, depending upon the availability of either USL or LSL respectively. Thus,

$$C_p^I(u,v) = \begin{cases} C_p^U(u,v), & \text{when only USL is available} \\ C_p^L(u,v), & \text{when only LSL is available} \end{cases} \quad (2)$$

We shall eventually observe that, since by definition, the mathematical formulations of $C_p^U(u,v)$ and $C_p^L(u,v)$ get changed only according to the availability of either USL or LSL, their statistical properties are similar. Therefore, we shall study the properties of $C_p^I(u,v)$, in general and discuss discuss about individual features of $C_p^U(u,v)$ and $C_p^L(u,v)$, wherever applicable.

Grau (2009) did not provide any formulation of 'k' and this makes the choice of 'k' rather subjective leaving room for favourable manipulation by the concerned stake holders. Latter, Chatterjee

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and Chakraborty (2012) have proposed a formulation of 'k' as

$$k = \frac{\text{Selling Price Per Item}}{\text{ALP}} \quad (3)$$

where, ALP denotes the average loss of profit per item due to deviation from 'T' towards the opposite side of the existing specification limit. Grau (2009) has also studied the distributional properties and expectations of the plug-in estimators of the member indices of $C_p^I(u, v)$. However, these expressions are complicated in nature and hence are unsuitable for potential future applications.

Moreover, since a process is a continuous flow of activities; while assessing its capability, often it is not justifiable to draw conclusion based on single sample information only. Even the conventional multiple sample estimation procedure for PCIs has a general tendency of smoothing out some important fluctuations in a process. Moreover, stability of a process does not always ensure its consistent capability over the period of time [Chatterjee and Chakraborty (2013b)]. Control charts, by definition, usually do not take into account the concepts like specification limits and target, which are of great importance from the view point of the acceptability of the produced items among the end users. As a result, even if, by using suitable control chart(s), a process is found to be stable, it may not imply that the process is performing satisfactorily. The process may suffer from decreased proximity of the process centering from the target or increased level of process variability with respect to the specification limits. These incidences are likely to be overlooked by the usual control charts for which the control limits are set based on the observed data and hence for a process, stability may indeed be attained - but at far the from satisfactory level. Thus, for a process with inconsistent capability, the PCI value based on the conventional single or multiple sample information may not reflect the actual process capability and they tend to smooth-out some important fluctuations in the observed values of the concerned quality characteristic [Chatterjee and Chakraborty (2013b)]. A possible solution to these problems is the use of process capability control chart of the concerned PCI based on the information from the corresponding $\overline{X} - R$ and

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$\overline{X} - S$ charts, which were already used for checking stability of the said process [refer Spiring (1995) and Chatterjee and Chakraborty (2013b)]. Morita et al. (2009) and Carot et al. (2013) have designed process capability control chart for the PCIs C_{pm} and C_{pmk}. Vännman and Albing also argued for using process capability plots for assessing capability of a process having unilateral specification limits.

In the present article, we have proposed more tractable forms of the statistical distributions and the expectations of these plug-in estimators. In particular, we have computed the minimum variance unbiased estimator (UMVUE) of C^I_{pk} based on single sample information. We have also computed the threshold value of C^I_p and have established exact relationship between C^I_{pk} and the proportion of non-conformance (PNC). These concepts have immense importance from the interpretability as well as application point of view of a PCI, but had hardly been explored in the literature.

Moreover, going by the analogy of Spiring (1995) and Chatterjee and Chakraborty (2013b), we have studied the distributional and inferential properties of the member indices of $C^U_p(u, v)$ and $C^L_p(u, v)$ based on information from the corresponding $\overline{X} - R$ and $\overline{X} - S$ charts and have designed the concerned process capability control charts as well. Through some numerical examples, we have also reinstated the importance of using such process capability control charts to have a vivid picture of the process performance throughout the entire production cycle.

In the following section, we have enlisted the notations which are used throughout this article. Section 3 contains some important statistical properties of $C^I_p(u, v)$, viz., threshold value of C^I_p and the relationship between C^I_p, C^I_{pk} and the proportion of non-conformance (PNC). In section 4, the distributional aspects along with the expressions for the expectations of the plug-in estimators of some member indices of $C^I_p(u, v)$, based on the single sample information, are discussed while section 5 deals with the same distributional aspects of these PCIs based on information from the corresponding $\overline{X} - R$ and $\overline{X} - S$ charts. Their process capability control charts are designed in section 6 followed by a numerical example in section 7. Finally, the article concludes in section 8 with a general note on the topics discussed in this paper.

2 List of Notations

1. U: Upper specification limit (USL);
2. L: Lower specification limit (LSL);
3. $d = \frac{U-L}{2}$;
4. $M = \frac{U+L}{2}$;
5. ‘T’ is the target;
6. X is a random variable characterizing the quality characteristic under consideration;
7. $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$;
8. $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$;
9. $k(> 1)$ quantifies the risk of deviation from ‘T’ in the direction opposite to the available specification limit with respect to ‘T’;
10. $A_U^* = \max\left\{(\mu - T), \frac{T-\mu}{k}\right\}$;
11. $A_L^* = \max\left\{\frac{\mu-T}{k}, (T - \mu)\right\}$;
12. ‘u’ and ‘v’ are two non-negative parameters;
13. $D_U = U - T$;
14. $D_L = T - L$;
15. $d^* = \min(D_U, D_L)$;

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16. $k^* = \max\left(\frac{D_U}{D_L}, \frac{D_L}{D_U}\right)$;

17. $R_U = \frac{\mu - T}{D_U}$;

18. $R_L = \frac{T - \mu}{D_L}$;

19. $\alpha_U = \max[\frac{\mu - T}{U - T}, \frac{T - \mu}{k(U - T)}]$;

20. $\delta_U = \frac{(U - T)\sqrt{n}}{\sigma}$;

21. $\delta_L = \frac{(T - L)\sqrt{n}}{\sigma}$;

22. $\delta_U^* = \frac{\mu - T}{\sigma} \times I_k^U$;

23. $I_k^U = \begin{cases} 1, & \text{for } \mu > T \\ -\frac{1}{k}, & \text{for } \mu < T \end{cases}$;

24. $\delta_L^* = \frac{\mu - T}{\sigma} \times I_k^L$;

25. $I_k^L = \begin{cases} -\frac{1}{k}, & \text{for } \mu > T \\ 1, & \text{for } \mu < T \end{cases}$;

26. $\delta_1 = n\left(\frac{\mu - T}{\sigma}\right)^2$;

27. $\chi^{*U(n)^2} = \frac{1}{n-1}\chi_{n-1}^2 + \frac{1}{n} I_k^{U^2} \chi_1^2(\delta_1)$ and $\delta_1 = n(\frac{\mu - T}{\sigma})^2$;

28. $\chi^{*L(n)^2} = \frac{1}{n-1}\chi_{n-1}^2 + \frac{1}{n} I_k^{L^2} \chi_1^2(\delta_1)$;

29. $b_{n-1} = \frac{\sqrt{\frac{2}{n-1}}\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)}$;

30. $d_2^* = \sqrt{d_2^2 + \frac{d_3^2}{m}}$;

31. $\nu = \frac{1}{-2 + 2\sqrt{1 + \frac{2d_3^2}{md_2^2}}}$;

32. $\delta_U^{(N)} = \frac{\sqrt{N}(U - T)}{\sigma}$;

33. $\delta_L^{(N)} = \frac{\sqrt{N}(T - L)}{\sigma}$.

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3 Some Useful Properties of $C_p^I(u, v)$

The very success of a process capability index lies in its acceptability among the practitioners. Hence, a PCI should be easy to formulate and unambiguously interpretable. Exact relationship with proportion of non-conformance and expression for the threshold value of the concerned PCI are two of the most important properties from the view point of the applicability of a PCI in practice.

3.1 Relationship of Proportion of Non-conformance with C_p^I and C_{pk}^I

The probability of producing an item, having the corresponding quality characteristic value beyond the pre-assigned specification limit(s), is known as the proportion of non-conformance (PNC). Thus, PNC or equivalently, the process yield ($= 1 - PNC$) is one of the major factors for measuring process performance. Since, PNC and PCI are the two parallel approaches of assessing the performance of a process, having exact relationship with PNC is considered to be an added advantage for a PCI.

Suppose P_{NC}^U denotes the probability of producing non-conforming items when the process is centered at 'T', i.e. $\mu = T$ and only USL exists. Similarly, P_{NC}^L is defined for the situation when only LSL exists. Then, $P_{NC}^U = \Phi(-3C_p^U)$ and $P_{NC}^L = \Phi(-3C_p^L)$ depending upon the availability of USL or LSL respectively [Grau (2009)].

However, the expressions for P_{NC}^U and P_{NC}^L are based on the assumption that the process is centered at 'T'. Thus, when $\mu \neq T$, P_{NC}^U and P_{NC}^L do not measure the actual PNC.

Since C_{pk}^U and C_{pk}^L are defined analogous to the yield based index C_{pk}, it is logical to expect that there would be some exact relationship between C_{pk}^U or $C_p^L(u, v)$ and the PNC, when $\mu \neq T$. Let us denote this PNC as $P_{NC}^{E(U)}$ or $P_{NC}^{E(L)}$, depending upon the availability of USL and LSL respectively. This relationship is established through the following theorem:

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Theorem 1:

$$P_{NC}^{E(U)} = \begin{cases} 1 - \Phi\left[3\{C_{pk}^{U} - (\frac{k+1}{k})R_U C_p^U\}\right] & \text{for } \mu < T \\ 1 - \Phi[3C_{pk}^{U}], \text{for } \mu > T \end{cases} \quad (4)$$

where, $R_U = \frac{\mu - T}{D_U}$ and $D_U = U - T$.

Proof: Before going into the actual proof of the theorem, let us consider a virtual LSL to a process having a quality characteristic of smaller the better type, such that $D_L = kD_U$, where, $D_L = T - L$. Under such circumstances, Grau (2009) has shown that

$$C_p^U(u, v) = C_p''(u, \frac{4v}{(1+k)^2}) \quad (5)$$

Here, $C_p''(u, v)$ denotes a superstructure of PCIs for asymmetric specification limits [Chen and Pearn (2001)].

In fact, since by definition $k > 1$, introduction of the virtual LSL converts the unilateral specification limit to asymmetric specification limits. While exploring the relationship between PNC and C_{pk}'', the PCI (analogous to C_{pk}) for asymmetric specification limits , Chatterjee and Chakraborty (2013a) have considered four different situations based on various possible positions of 'T' with respect to U, L and μ. Among these, only two are relevant for smaller the better type quality characteristic, viz., $\mu < T$ and $\mu > T$ (since here, $d^* = \min(D_U, D_L) = D_U$ always).

Now, let us define $k^* = \max(\frac{D_U}{D_L}, \frac{D_L}{D_U}) = k$ and $R_L = \frac{T-\mu}{D_L} = \frac{T-\mu}{kD_U}$. Also, from equation (5), $C_{pk}^U = C_p^U(1, 0) = C_p''(1, 0) = C_{pk}''$ and $C_p^U = C_p''$. Thus, $P_{NC}^{E(U)}$ can be formulated as

$$\begin{aligned} P_{NC}^{E(U)} &= 1 - P(X < U | X \sim N(\mu, \sigma^2)) \\ &= 1 - \Phi\left[\frac{D_U}{\sigma}(1 - R_U)\right] \end{aligned} \quad (6)$$

<u>*Case I: $\mu < T$*</u> Here, following Chatterjee and Chakraborty (2013a), we have,

$$\begin{aligned} P_{NC}^{E(U)} &= 1 - \Phi\left[\frac{D_U}{\sigma}(1 - R_U)\right] \\ &= 1 - \Phi\left\{3[C_{pk}'' + (1 + k^*)\, R_L C_p'']\right\} \\ &= 1 - \Phi\left\{3\, [C_{pk}^U - \left(\frac{1+k}{k}\right) R_U C_p^U]\right\} \end{aligned} \quad (7)$$

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Case II: $\mu > T$ Here, $\frac{D_U}{\sigma}(1 - R_U) = 3C''_{pk} = 3C^U_{pk}$, [from equation (5) and Chatterjee and Chakraborty (2013a)]. Hence, from equation (6),

$$P^{E(U)}_{NC} = 1 - \Phi[3C^U_{pk}] \tag{8}$$

Thus, combining equations (7) and (8) Theorem 1 follows.

Note that for a quality characteristic of smaller the better type, the situation viz., $\mu < T$ is more desirable than that of $\mu > T$ as the former signifies lesser value of the quality characteristic on an average. Since 'Φ' is an increasing function, it is easy to see that $P^{E(U)}_{NC}|_{\mu<T} < P^{E(U)}_{NC}|_{\mu>T}$ which should ideally be the case and this validates our formulation. However, for $\mu < T$, P^U_{NC} does not ensure providing minimum observable PNC. In fact $\mu < T$ implies that on an average, values of the quality characteristics are less than 'T' indicating increase in the overall quality level and hence, this type of deviation from 'T' can not be considered as added contribution to PNC, for smaller the better type of quality characteristic. Moreover, although the computations involved in theorem 1 was based on the virtual LSL, the final form of the exact relationship between $P^{E(U)}_{NC}$ and C^U_{pk} is free from that and this is highly desirable.

Similarly, it can be shown that for quality characteristics of larger the better type, $P^{E(L)}_{NC}$ can be formulated as

$$P^{E(L)}_{NC} = \begin{cases} 1 - \Phi[3C^L_{pk}] \ \text{ for } \mu < T \\ 1 - \Phi\left[3\{C^L_{pk} - (\frac{k+1}{k})R_L C^L_p\}\right], \text{for } \mu > T \end{cases} \tag{9}$$

with $P^{E(L)}_{NC}|_{\mu>T} < P^{E(L)}_{NC}|_{\mu<T}$ as desired. Also, for $\mu > T$, P^L_{NC} does not ensure providing minimum observable PNC.

Thus, similar to the cases of both the symmetric and asymmetric specification limits, for unilateral specification limits also, when $\mu \neq T$, the PNC can be expressed in terms of C^U_p and C^U_{pk}.

In this context, for $\mu \neq T$, Grau (2012) has developed an expression for the upper bound of PNC; while our formulation provides the exact relationship. Moreover, although $C^U_p(u, v)$ can

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be expressed in terms of $C''_p(u, v)$ [refer Grau (2009)]; while computing $P_{NC}^{E(U)}$, one needs to consider only the situations where the values of the quality characteristic exceeds USL as LSL is not available here. Hence direct application of the formulation of PNC in terms of C''_p and C''_{pk} [refer Chatterjee and Chakraborty (2013a)] is not appropriate here. In fact, the said formulation will give the upper bound of the PNC for unilateral specification limits similar to Grau's (2012) formulation, when $\mu \neq T$. For example, Grau (2012) has calculated the upper bound of NCPPM for $C_{pk}^U = 1$ as 1350 and this is exactly the same value of NCPPM for $C''_{pk} = 1$ [refer Chatterjee and Chakraborty (2013a)]. This argument is valid for quality characteristics of larger the better type as well.

Result1: $P_{NC}^{E(U)} \uparrow k$, for $\mu < T$ and for fixed C_p^U and C_{pk}^U values.

Proof: Since C_{pk}^U is independent of 'k' for $\mu > T$, we have to consider the case of $\mu < T$ only. Here, $R_U = \frac{\mu - T}{D_U} < 0$. Also, by definition, $k > 1$. Hence, $3\left[C_{pk}^U - (1 + \frac{1}{k})R_U C_p^U\right] \downarrow k$. Thus, '$\Phi$' being an increasing function, from equation (7) result 1 follows. Similar result also holds good for $P_{NC}^{E(L)}$ with $\mu > T$.

The result can be logically explained from the definition of 'k' as well. Since in a competitive market, selling price can not be changed easily, the numerator of equation (3) is assumed to be fixed and hence 'k' increases in inverse proportion with the average loss of profit due to deviation from T towards left (when USL is available). Now, one possible reason for decrease in the denominator of equation (3) may be that, for most of the sample observations, the values of the concerned quality characteristic deviate from target towards USL increasing the possibility of producing non-conforming items. This gives a logical foundation to result 1.

From Result 1, $R_U C_p < 0$ for $\mu < T$. Now, for a unilateral process with USL, it is always desirable to have $D_U > (T - \mu)$ and hence, $-1 < R_U C_p < 0$. In fact, $R_U C_p$ should be as close to 0 as possible. Also, often for a process with $\mu \neq T$, the PNC values are found to be very small and hence they are expressed in terms of the non-conforming parts per million ($NCPPM^E$) which is 10^6 times the PNC. In Table 1, the $NCPPM^E$ values are tabulated for different values of 'k' and

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C^U_{pk} (or, C^L_{pk}) when $\mu > T$ (or, $\mu < T$).

TABLE 1 SHOULD BE ABOUT HERE

Also, Table 2 gives the $NCPPM^E$ values for different values of 'k' and C^U_{pk} (or, C^L_{pk}) when $\mu < T$ (or, $\mu > T$) with $R_U C_p = -0.3$. For other values of $R_U C_p$, the corresponding values of $NCPPM^E$ can be computed using the equation (4) or (9) depending upon the available specification limit.

TABLE 2 SHOULD BE ABOUT HERE

Table 2 shows that for fixed 'k', $NCPPM^E$ decreases with the increase in the C^I_{pk} value. This is quite logical as higher the value of C^I_{pk}, better is the process. Moreover, for any fixed 'k', the value of $NCPPM^E$ corresponding to each C^I_{pk} in Table 2 is much less than the $NCPPM^E$ value for the same value of C^I_{pk} tabulated in Table 1. This is similar to our observation in Theorem 1.

3.2 Threshold Value of C^I_p

The concept of threshold value plays a prime role in the context of the interpretation and practical application of a PCI. A threshold value is such a value of a PCI that a process with the PCI value higher than this threshold value is considered to be capable; while for an incapable process, the observed PCI value is less than the threshold value. Following the general convention, the threshold value is generally computed for the potential PCI; since a process which is not even potentially capable will, in all likelihood, have very poor capability level under the given specification criteria. Unlike C_{PU} (or, C_{PL}), C^U_p (or, C^L_p) measures the potential capability of a process as, by definition, it is independent of μ. Hence, the expressions for the threshold values of C^U_p and C^L_p need to be explored here.

Going by the usual convention, the threshold value of C^U_p should be 1. However, if similar logic, as in case of C_p is used, then, the threshold value of C^U_p should be that value for which the voice of the customer (measured by specification spread) coincides with the voice of the process (reflected by the process variation).

Let us first consider a quality characteristic of smaller the better type. Also, for ease of representation, we consider, for the time being, existence of a LSL with $D_L = kD_U$. This is similar to our discussion in section 3.1. Here, the specification spread will be $U - L = (k+1)(U - T)$. Also, since we have already assumed that the quality characteristic under consideration follows $N(\mu, \sigma^2)$, the process spread will be 6σ. Thus,

$$\frac{\text{Voice of the Customer}}{\text{Voice of the Process}} = \left(\frac{1+k}{2}\right) C_p^U$$

Similar is the case for C_p^L as well. Hence, the threshold value of C_p^I will be

$$C_p^{I(T)} = \frac{2}{1+k} \tag{10}$$

The threshold values of C_p^I for various values of 'k' are tabulated in Table 3.

TABLE 3 SHOULD BE ABOUT HERE

The following observations can be made regarding these threshold values:

1. Unlike the case of symmetric specification limits and contradicting the usual convention, for unilateral specification limits, the threshold value is not unique; rather it varies as a function of 'k'.

2. $C_p^{I(T)} \downarrow k$ with $C_p^{I(T)} \rightarrow 1$ for $k \rightarrow 1$ and for $k \rightarrow \infty$, $C_p^{I(T)} \rightarrow 0$.

3. For $k = 1$ we have $D_U = D_L$ which indicates symmetry of specification limits with respect to T and then $C_p^{I(T)} = 1$, the value which is mostly considered as the threshold value in practice.

4. Since by definition $k > 1$, $C_p^{I(T)} < 1$. Thus considering 1 as the threshold value of $C_p^{I(T)}$ underestimates the potential capability of a process. In this context, while proposing minimum desirable (threshold) values of the PCIs for various situations for both the bi-lateral and

unilateral specification limits, Montgomery (2010) recommended lower threshold values for processes with unilateral specification limits, compared to those with bi-lateral specification limits, irrespective of the nature of the process.

5. Similar story is revealed from the viewpoint of proportion of non-conformance as well. The PNC corresponding to $C_p = 1$ is 0.0027%, while PNC corresponding to $C_p^I = 1$ is 0.00135%. Thus, suppose for a process with $\mu \neq T$, the PNC is p' such that $0.0000135 < p' < 0.000027$. Then consideration of $C_p^I = 1$ as the threshold value will consider the process to be potentially incapable, while the actual situation may not be that bad.

4 Distributional Properties of the Plug-in Estimators of $C_p^I(u, v)$ for $u = 0, 1$ and $v = 0, 1$ Based on Single Sample Information

Although PCIs are primarily defined for application in industries, their definitions involve the parameters of the concerned quality characteristics and hence are often unobservable. The common industrial practice is to calculate the values of the plug-in estimators of the actual PCIs based on the available sample information and to decide about the capability level of the said process based on that value. So the statistical properties of the plug-in estimators of PCIs need to be studied. Grau (2009) have studied the expressions for the underlying statistical distributions and the expectations of the plug-in estimators of $C_p^I(u, v)$ for $u = 0, 1$ and $v = 0, 1$. However, his formulations (in particular for the PCIs excluding C_p^I) are a bit complicated and hence not suitable for prospective future applications. Here, our objective is to explore more tractable forms of the said distributions and expectations.

Suppose a sample of size ‘n’ is drawn from a process and X_i is the value of the concerned quality characteristic for the i^{th} sample observation, for i = 1(1)n such that $X_i \sim N(\mu, \sigma^2)$. From equation

(1), C_{pk}^U can be defined as

$$C_{pk}^U = (1-\alpha_U)C_p^U = \begin{cases} C_{PU}, & \text{for } \mu \geq T \\ (\frac{k+1}{k})C_p^U - \frac{1}{k}C_{PU}, & \text{for } \mu < T \end{cases} \quad (11)$$

where, $\alpha_U = \max[\frac{\mu-T}{U-T}, \frac{T-\mu}{k(U-T)}]$ and $C_p^U = \frac{U-T}{3\sigma}$. Accordingly, the corresponding plug-in (natural) estimator will be,

$$\widehat{C}_{pk}^U = \begin{cases} \widehat{C}_{PU}, & \text{for } \mu \geq T \\ (\frac{k+1}{k})\widehat{C}_p^U - \frac{1}{k}\widehat{C}_{PU}, & \text{for } \mu < T \end{cases} \quad (12)$$

with $\widehat{C}_p^U = \frac{U-T}{3S}$ and $\widehat{C}_{PU} = \frac{U-\overline{X}}{3S}$, where, $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ are the sample mean and variance for the said quality characteristic respectively.

The statistical distribution of $\widehat{C}_{pk}^U$ and the expression for the corresponding unbiased estimator are derived in the following theorem:

Theorem 2:

$$\widehat{C}_{pk}^U \sim \begin{cases} \frac{1}{3\sqrt{n}} t_{n-1}(\delta_U), & \text{for } \mu \geq T \\ \frac{(k+1)(U-T)\sqrt{n-1}}{3k\sigma} \times \chi_{n-1}^{-1} - \frac{1}{3k\sqrt{n}} \times t_{n-1}(\delta_U), & \text{for } \mu < T \end{cases} \quad (13)$$

where, $t_{n-1}(\delta_U)$ denotes the non-central t-distribution with $(n-1)$ degrees of freedom with the corresponding non-centrality parameter $\delta_U = \frac{(U-T)\sqrt{n}}{\sigma}$ and χ_{n-1}^{-1} denotes inverse central chi-distribution with $(n-1)$ degrees of freedom.

Also,

$$\widetilde{C}_{pk}^U = \begin{cases} b_{n-1} \times \left(\frac{U-\overline{X}}{3S}\right), & \text{for } \mu \geq T \\ b_{n-1} \times \left[\frac{(k+1)(U-T)}{3kS} - \frac{U-\overline{X}}{3kS}\right], & \text{for } \mu < T \end{cases} \quad (14)$$

is the UMVUE of C_{pk}^U corresponding to the plug-in estimator $\widehat{C}_{pk}^U$, given in equation (12).

Proof: For $\mu \geq T$, $\widehat{C}^U_{pk} = \widehat{C}_{PU}$ for which expressions for the underlying statistical distribution and the UMVUE are already available in literature [see Chatterjee and Chakraborty (2012)]. Therefore, only for $\mu < T$, the distribution and UMVUE of $\widehat{C}^U_{pk}$ need to be explored. Now, $\widehat{C}^U_p \sim \frac{(U-T)\sqrt{n-1}}{3\sigma}\chi^{-1}_{n-1}$ [Grau (2009)]. Thus, from equations (11) and (12) and using the distributions of $\widehat{C}^U_p$ and $\widehat{C}_{PU}$, the distribution of $\widehat{C}^U_{pk}$ is obtained as in (13).

Again, from (12), when $\mu < T$,

$$\begin{aligned} E[\widehat{C}^U_{pk}] &= \left(\frac{k+1}{k}\right)E(\widehat{C}^U_p) - \frac{1}{k}E(\widehat{C}_{PU}) \\ &= \sqrt{\frac{n-1}{2}} \times \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} C^U_{pk} \end{aligned}$$

Thus, when $\mu < T$, $\widetilde{C}^U_{pk} = b_{n-1}\widehat{C}^U_{pk} = b_{n-1}\left[\left(\frac{k+1}{k}\right)\widehat{C}^U_p - \frac{1}{k}\widehat{C}_{PU}\right]$ is an unbiased estimator of C^U_{pk}. Hence, combining the two situations, viz., $\mu \geq T$ and $\mu < T$, (14) indeed gives the expression for an unbiased estimator of C^U_{pk}.

Observe that under the assumption of normality of the quality characteristic, $(\overline{X}, S^2)$ are jointly complete sufficient statistics for (μ, σ^2). Thus, using Rao - Blackwell theorem [see Casella and Berger (2007)], $\widetilde{C}^U_{pk}$ is the UMVUE of C^U_{pk} and this completes the proof of Theorem 2.

Interestingly, the property of C^U_{pk}, that it is independent of 'k' for shift of μ from 'T' towards the existing specification limit (here USL), is also retained in these expressions for the distribution and the plug-in estimator of C^U_{pk}.

Similarly, for quality characteristics of higher the better type,

$$\widehat{C}^L_{pk} \sim \begin{cases} \frac{(k+1)(T-L)\sqrt{n-1}}{3k\sigma} \times \chi^{-1}_{n-1} - \frac{1}{3k\sqrt{n}} \times t_{n-1}(\delta_L), & \text{for } \mu \geq T \\ \frac{1}{3\sqrt{n}} t_{n-1}(\delta_L), & \text{for } \mu < T \end{cases}$$

where, $\delta_L = \frac{\sqrt{n}(T-L)}{\sigma}$. Also,

$$\widetilde{C}^L_{pk} = \begin{cases} b_{n-1}\left[\frac{(k+1)(T-L)}{3kS} - \frac{\overline{X}-L}{3kS}\right], & \text{for } \mu \geq T \\ b_{n-1} \times \left[\frac{\overline{X}-L}{3S}\right], & \text{for } \mu < T \end{cases}$$

is the UMVUE of C_{pk}^{L} corresponding to the plug-in estimator $\widehat{C}_{pk}^{L}$.

Let us now re-define C_{pm}^{U} as $C_{pm}^{U} = (1 + \delta_{U}^{*2})^{-\frac{1}{2}} \times C_{p}^{U}$, where, $\delta_{U}^{*} = \frac{\mu - T}{\sigma} \times I_{k}^{U}$. Then,

$$\widehat{C}_{pm}^{U} \sim \frac{U - T}{3\sigma} \times \left[\chi^{*U(n)}\right]^{-1} \tag{15}$$

Similarly, for $C_{pm}^{L} = (1 + \delta_{L}^{*2})^{-\frac{1}{2}} \times C_{p}^{L}$, we have, $\widehat{C}_{pm}^{L} \sim \frac{T-L}{3\sigma} \times \left[\chi^{*L(n)}\right]^{-1}$, where, $\delta_{L}^{*} = \frac{\mu - T}{\sigma} \times I_{k}^{L}$.

However, the linear combination of a central and a non-central chi-square distribution do not follow any standard distribution an hence, $\chi^{*U(n)^2}$ and $\chi^{*L(n)^2}$ can not be simplified further.

Again, from equation (1) and following Grau (2009), C_{pmk}^{U} can be defined as $C_{pmk}^{U} = C_{pm}^{U} - \alpha_U \times C_{pm}^{U}$. The corresponding plug-in estimator viz., $\widehat{C}_{pmk}^{U}$ may be obtained by replacing μ and σ by $\overline{X}$ and S respectively, in the definition of C_{pmk}^{U}. The following theorem gives the distribution of $\widehat{C}_{pmk}^{U}$.

Theorem 3: $\widehat{C}_{pmk}^{U} \sim \frac{U-T}{3\sigma} \times \left[\chi^{*U(n)}\right]^{-1} - \frac{1}{3\sqrt{1+\frac{n}{I_k^{U2}} \times F_{1,n-1}(\delta_1)}}$, where, $F_{1,n-1}(\delta_1)$ denotes the non-central F distribution with 1 and (n - 1) degrees of freedom and δ_1 is the corresponding non-centrality parameter.

Proof: <u>*Case I* ($\mu \geq T$)*:*</u> Here,

$$\begin{aligned}
\widehat{C}_{pmk}^{U} &= \frac{U - T}{3\sqrt{S^2 + (\overline{X} - T)^2}} - \frac{1}{3\sqrt{1 + (\frac{S}{\overline{X}-T})^2}} \\
&= \frac{U - T}{3\sigma} \times \frac{1}{\sqrt{(\frac{S}{\sigma})^2 + (\frac{\overline{X}-T}{\sigma})^2}} - \frac{1}{3\sqrt{1 + \left(\frac{S}{\overline{X}-T}\right)^2}}
\end{aligned} \tag{16}$$

Now, $(\frac{S}{\sigma})^2 + (\frac{\overline{X}-T}{\sigma})^2 \sim \frac{1}{n-1}\chi_{n-1}^2 + \frac{1}{n}\chi_1^2(\delta_1)$ and $(\frac{S}{\overline{X}-T})^2 \sim nF_{1,n-1}(\delta_1)$, since, S and $\overline{X}$ are independently distributed. Hence, from equation (16),

$$\widehat{C}_{pmk}^{U} \sim \frac{U - T}{3\sigma} \times \frac{1}{\sqrt{\frac{1}{n-1}\chi_{n-1}^2 + \frac{1}{n}\chi_1^2(\delta_1)}} - \frac{1}{3\sqrt{1 + nF_{1,n-1}(\delta_1)}} \tag{17}$$

<u>*Case II* ($\mu < T$)*:*</u> Here,

$$\widehat{C}_{pmk}^{U} = \frac{U - T}{3\sigma} \times \frac{1}{\sqrt{(\frac{S}{\sigma})^2 + \frac{1}{k^2}(\frac{\overline{X}-T}{\sigma})^2}} - \frac{1}{3\sqrt{1 + k^2\left(\frac{S}{\overline{X}-T}\right)^2}}$$

and hence,

$$\widehat{C}^{U}_{pmk} \sim \frac{U-T}{3\sigma\sqrt{\frac{1}{n-1}\chi^2_{n-1} + \frac{1}{nk^2}\chi^2_1(\delta_1)}} - \frac{1}{3\sqrt{1+nk^2F_{1,n-1}(\delta_1)}} \tag{18}$$

Combining equations (17) and (18), theorem 3 follows.

Similarly, $\widehat{C}^{L}_{pmk} \sim \frac{T-L}{3\sigma\sqrt{\frac{1}{n-1}\chi^2_{n-1}+\frac{I_k^{L2}}{n}\chi^2_1(\delta_1)}} - \frac{1}{3\sqrt{1+\frac{n}{I_k^{L2}}F_{1,n-1}(\delta_1)}}$.

5 Distributions and Expectations of the Plug-in Estimators of $C_p^I(u,v)$ for $u = 0,1$ and $v = 0,1$ Based on Control Chart Information

Suppose from a process, 'm' samples are drawn each of which is of size 'n'. So the total number of sample observations is $N = mn$. Let X_{ij} be the measured value of the quality characteristic corresponding to the j^{th} observation of the i^{th} sample, for $i = 1(1)m$, $j = 1(1)n$, such that $X_{ij} \sim N(\mu, \sigma^2)$. Also suppose, before computing the values of the PCIs, $\overline{X} - R$ or $\overline{X} - S$ charts are used to check and establish stability of the process. Then, following Spiring's (1995) approach, while defining the plug-in estimators of the member indices of $C_p^I(u,v)$, the parameters of the quality characteristics, viz., μ and σ, should be replaced respectively by $\overline{\overline{X}}$ and $\overline{R}/d_2$ (if $\overline{X} - R$ chart information is used) or by $\overline{\overline{X}}$ and $\overline{S}/c_4$ (if $\overline{X} - S$ chart information is used) in equation (1), where, c_4 and d_2 are two well known constants of control charts. Such estimation policy is more economical than the classical one, in a sense that in the classical estimation, while checking stability of a process, samples are drawn from it to construct suitable control chart. These control charts provide a set of estimates for μ and σ. However, these estimates are never used in the subsequent stages of the performance assessment of the process. Rather, while computing PCIs, fresh samples are drawn for estimating the same parameters. This increases the sampling cost especially for processes requiring destructive tests or where the cost of production is exorbitant

[Chatterjee and Chakraborty (2013b)].

5.1 Distributions and Expectations of $C_p^I(u,v)$ for $u = 0, 1$ and $v = 0, 1$ Based on Information from $\overline{X} - R$ Charts

From equation (1), C_p^U can be defined as $C_p^U = (U-T)/3\sigma$ and following Spiring (1995), the corresponding plug-in estimator, based on information from $\overline{X} - R$ Charts, will be $\widehat{C}_p^{U(R)} = \frac{d_2(U-T)}{3\overline{R}}$.

Woodall and Montgomery (2000) have shown that $\frac{\overline{R}}{\sigma} \sim d_2^* \frac{\chi_\nu}{\sqrt{\nu}}$, where, $d_2^* = \sqrt{d_2^2 + \frac{d_3^2}{m}}$ and $\nu = \frac{1}{-2+2\sqrt{1+\frac{2d_3^2}{md_2^2}}}$, [Kuo (2010)]. Thus,

$$\widehat{C}_p^{U(R)} \sim \frac{d_2(U-T)\sqrt{\nu}}{3d_2^*\sigma} \times \chi_\nu^{-1} \tag{19}$$

Also, $\widetilde{C}_p^{U(R)} = b_\nu \frac{d_2^*}{d_2} \widehat{C}_p^{U(R)}$ is the unbiased estimator of C_p^U based on $\overline{X} - R$ chart information, where, $b_\nu = \frac{\sqrt{\frac{2}{\nu}}\,\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu-1}{2}\right)}$. Similarly, $\widehat{C}_p^{L(R)} \sim \frac{d_2(T-L)\sqrt{\nu}}{3d_2^*\sigma} \times \chi_\nu^{-1}$ and $\widetilde{C}_p^{L(R)} = b_\nu \frac{d_2^*}{d_2} \widehat{C}_p^{L(R)}$ is the unbiased estimator of C_p^L, where, $\widehat{C}_p^{L(R)} = \frac{d_2(T-L)}{3\overline{R}}$.

We now derive the expressions for the underlying statistical distributions as well as the expectations of the plug-in estimators of C_{pk}^U and C_{pk}^L with information from $\overline{X} - R$ chart. Note that corresponding to the definition of C_{pk}^U, as given in equation (11), the expression for the plug-in estimator of C_{pk}^U, based on the information from the respective $\overline{X} - R$ charts will be,

$$\widehat{C}_{pk}^{U(R)} = \begin{cases} \widehat{C}_{PU}^{(R)}, & \text{for } \mu \geq T \\ \left(\frac{k+1}{k}\right)\widehat{C}_p^{U(R)} - \frac{1}{k}\widehat{C}_{PU}^{(R)}, & \text{for } \mu < T \end{cases} \tag{20}$$

The expression for the statistical distribution of $\widehat{C}_{PU}^{(R)}$ is derived in theorem 4 below.

Theorem 4:

$$\widehat{C}_{pk}^{U(R)} \sim \begin{cases} \frac{d_2}{3d_2^*\sqrt{N}} \times t_\nu(\delta_U^{(N)}), & \text{for } \mu \geq T \\ \left[\frac{(k+1)(U-T)d_2\sqrt{\nu}}{kd_2^*\sigma}\right] \times \chi_\nu^{-1} - \frac{d_2}{3d_2^* k\sqrt{N}} \times t_\nu(\delta_U^{(N)}), & \text{for } \mu < T \end{cases} \tag{21}$$

Also, $\widehat{C}_{pk}^{U(R)} = \frac{d_2^*}{d_2} \times b_\nu \times \widehat{C}_{pk}^{U(R)}$ is an unbiased estimator of C_{pk}^U.

Proof: <u>*Case I ($\mu \geq T$):*</u> Here, $\widehat{C}_{pk}^{U(R)} = \widehat{C}_{PU}^{(R)} = \frac{d_2(U-\overline{\overline{X}})}{3\overline{R}}$ [from equation (20)]. Hence, following Chatterjee and Chakraborty (2013b),

$$\widehat{C}_{pk}^{U(R)} = \widehat{C}_{PU}^{(R)} \sim \frac{d_2}{3d_2^* \sqrt{N}} \times t_\nu(\delta_U^{(N)}) \tag{22}$$

and $\widehat{C}_{pk}^{U(R)} = \frac{d_2^*}{d_2} \times b_\nu \times \widehat{C}_{pk}^{U(R)}$ is an unbiased estimator of C_{pk}^U when information gathered from $\overline{X} - R$ chart is used for parameter estimation.

<u>*Case II ($\mu < T$):*</u> Here, from equations (19), (20) and (22),

$$\widehat{C}_{pk}^{U(R)} \sim \left[\frac{(k+1)(U-T)d_2\sqrt{\nu}}{3kd_2^*\sigma}\right] \times \chi_\nu^{-1} - \frac{d_2}{3d_2^* k\sqrt{N}} t_\nu(\delta_U^{(N)}) \tag{23}$$

Thus, combining equations (22) and (23), the expression for the underlying statistical distribution of $\widehat{C}_{pk}^{U(R)}$, as given in equation (21) can be obtained.

Again, for $\mu < T$,

$$\begin{aligned} E[\widehat{C}_{pk}^{U(R)}] &= \frac{d_2}{d_2^*} \times b_\nu^{-1} \left[\left(\frac{k+1}{k}\right) \times C_p^U - \frac{1}{k} C_{PU}\right] \\ &= \frac{d_2}{d_2^*} \times b_\nu^{-1} \times C_{pk}^U \end{aligned}$$

Thus, $\widehat{C}_{pk}^{U(R)} = \frac{d_2^*}{d_2} \times b_\nu \times \widehat{C}_{pk}^{U(R)}$ is an unbiased estimator of C_{pk}^U.

Hence, combining the cases I and II, theorem 4 follows.

Similarly,

$$\widehat{C}_{pk}^{L(R)} \sim \begin{cases} \left[\frac{(k+1)(T-L)d_2\sqrt{\nu}}{3d_2^*\sigma}\right] \times \chi_\nu^{-1} - \frac{d_2}{3d_2^* k\sqrt{N}} \times t_\nu(\delta_L^{(N)}), & \text{for } \mu \geq T \\ \frac{d_2}{3d_2^*\sqrt{N}} \times t_\nu(\delta_L^{(N)}), & \text{for } \mu < T \end{cases}$$

Also, $\widehat{C}_{pk}^{L(R)} = \frac{d_2^*}{d_2} \times b_\nu \times \widehat{C}_{pk}^{L(R)}$ is an unbiased estimator of C_{pk}^L.

Again, the plug-in estimator of C_{pm}^U, based on the information from $\overline{X} - R$ chart, can be defined as $\widehat{C}_{pm}^{U(R)} = (1 + \widehat{\delta}_U^{*(R)^2})^{-\frac{1}{2}} \times \widehat{C}_p^{U(R)}$, where, $\widehat{\delta}_U^{*(R)}$ is defined as $\widehat{\delta}_U^{(R)} = \frac{d_2(\overline{\overline{X}}-T)}{\overline{R}} \times I_k^U$. Therefore,

$$\widehat{C}_{pm}^{U(R)} \sim \frac{U-T}{3\sigma} \times \left[\chi^{*U(R)}\right]^{-1} \tag{24}$$

where, $\chi^{*U(R)^2} = \frac{1}{\nu}\left(1 + \frac{d_3^2}{md_2^2}\right)\chi_\nu^2 + \frac{1}{N} I_k^{U^2} \chi_1^2(\delta_1^{(N)^2})$ and $\delta_1^{(N)^2} = N(\frac{\mu - T}{\sigma})^2$. The expression for the statistical distribution of C_{pm}^L can be obtained similarly.

Again, as obtained from theorem 3, the statistical distribution of $\widehat{C}_{pmk}^{U(R)}$ will be,

$$\widehat{C}_{pmk}^{U(R)} \sim \frac{U - T}{3\sigma \sqrt{\chi^{*U(R)^2}}} - \frac{1}{3\sqrt{1 + N\left(\frac{d_2^*}{d_2 I_k^U}\right)^2 \times F_{1,\nu}(\delta_1^{(N)^2})}} \tag{25}$$

where, $\widehat{C}_{pmk}^{U(R)}$ is the plug-in estimator of C_{pmk}^U based on information from $\overline{X} - R$ charts. The distribution of $\widehat{C}_{pmk}^{L(R)}$ can now be computed accordingly.

5.2 Distributions and Expectations of $C_p^I(u, v)$ for $u = 0, 1$ and $v = 0, 1$ Based on Information from $\overline{X} - S$ Charts

In the context of statistical quality control, range is the most widely used measure of dispersion due to its computational simplicity as well as ease of interpretation. Consequently, $\overline{X} - R$ charts are often used to check and establish stability of a process. However, sometimes, range fails to measure the dispersion of a process efficiently. This can happen when either the sample size is moderately large, say, $n > 10$ or the sample size is not constant. Under such circumstances, $\overline{X} - S$ charts are to be used instead of $\overline{X} - R$ charts.

Following Spiring's (1995) approach, the plug-in estimator of C_p^U based on information from the corresponding $\overline{X} - S$ chart is, $\widehat{C}_p^{U(S)} = \frac{c_4(U-T)}{3\overline{S}}$. Now, $(\frac{\overline{S}}{\sigma})^2 \sim \frac{1}{m(N-m)}\chi^2_{m(N-m)}$ [see Chatterjee and Chakraborty (2013b)] and hence,

$$\widehat{C}_p^{U(S)} \sim c_4 \sqrt{m(N-m)} \times \frac{U - T}{3\sigma} \times \chi^{-1}_{m(N-m)} \tag{26}$$

Also, $\widetilde{C}_p^{U(S)} = \frac{b_{m(N-m)}}{c_4} \times \widehat{C}_p^{U(S)}$ is an unbiased estimator of C_p^U.

Moreover, since under the assumption of normality of the distribution of the quality characteristic, $(\overline{\overline{X}}, \overline{S}^2)$ are complete sufficient statistics for (μ, σ^2), following Rao-Blackwell theorem, $\widetilde{C}_p^{U(S)}$ is the UMVUE of C_p^U. However, this was not the case with $\widetilde{C}_p^{U(R)}$ as $\overline{R}^2$ is not complete sufficient statistic for σ^2 and hence $\widetilde{C}_p^{U(R)}$ can not be considered as the UMVUE of C_p^U.

Similarly, $\widetilde{C}_p^{L(S)} \sim c_4 \sqrt{m(N-m)} \times \frac{T-L}{3\sigma} \times \chi^{-1}_{m(N-m)}$ and $\widetilde{C}_p^{L(S)} = \frac{b_{m(N-m)}}{c_4} \times \widetilde{C}_p^{L(S)}$ is the UMVUE of C_p^L, where, $\widetilde{C}_p^{L(S)} = \frac{c_4(T-L)}{3\overline{S}}$.

Again, corresponding to the definition of C_{pk}^U, as given in equation (11), the expression for the plug-in estimator, based on the information from the respective $\overline{X}-S$ charts, will be,

$$\widetilde{C}_{pk}^{U(S)} = \begin{cases} \widetilde{C}_{PU}^{(S)}, & \text{for } \mu \geq T \\ (\frac{k+1}{k})\widetilde{C}_p^{U(S)} - \frac{1}{k}\widetilde{C}_{PU}^{(S)}, & \text{for } \mu < T \end{cases} \tag{27}$$

Theorem 5:

$$\widetilde{C}_{pk}^{U(S)} \sim \begin{cases} \frac{c_4}{3\sqrt{N}} \times t_{m(N-m)}(\delta_U^{(N)}), & \text{for } \mu \geq T \\ \left[\frac{(k+1)(U-T)c_4\sqrt{m(N-m)}}{3k\sigma}\right] \times \chi^{-1}_{m(N-m)} - \frac{c_4}{3k\sqrt{N}} \times t_{m(N-m)}(\delta_U^{(N)}), & \text{for } \mu < T \end{cases} \tag{28}$$

Also, $\widetilde{C}_{pk}^{U(S)} = \frac{b_{m(N-m)}}{c_4} \times \widetilde{C}_{pk}^{U(S)}$ is the UMVUE of C_{pk}^U.

Proof: <u>*Case I ($\mu \geq T$):*</u> Here, $\widetilde{C}_{pk}^{U(S)} = \widetilde{C}_{PU}^{(S)} = \frac{c_4(U-\overline{\overline{X}})}{3\overline{S}}$ [from equation (27)]. Hence, following Chatterjee and Chakraborty (2013b),

$$\widetilde{C}_{pk}^{U(S)} = \widetilde{C}_{PU}^{(S)} \sim \frac{c_4}{3\sqrt{N}} \times t_{m(N-m)}(\delta_U^{(N)}) \tag{29}$$

and $\widetilde{C}_{pk}^{U(S)} = \frac{b_{m(N-m)}}{c_4} \times \widetilde{C}_{pk}^{U(S)}$ is the UMVUE C_{pk}^U when information gathered from $\overline{X}-S$ chart is used for parameter estimation.

<u>*Case II ($\mu < T$):*</u> Here, from equations (26), (27) and (29),

$$\widetilde{C}_{pk}^{U(S)} \sim \left(\frac{k+1}{k}\right) c_4 \sqrt{m(N-m)} \times \frac{U-T}{3\sigma} \times \chi^{-1}_{m(N-m)} - \frac{c_4}{3k\sqrt{N}} \times t_{m(N-m)}(\delta_U^{(N)}) \tag{30}$$

Hence, the distribution of $\widetilde{C}_{pk}^{U(S)}$ can be obtained by combining equations (29) and (30).

Also, for $\mu < T$,

$$E[\widetilde{C}_{pk}^{U(S)}] = \frac{c_4}{b_{m(N-m)}} \times \left[\left(\frac{k+1}{k}\right) \times C_p^U - \frac{1}{k} \times C_{PU}\right]$$

$$= \frac{c_4}{b_{m(N-m)}} \times C_{pk}^{U}$$

Thus, $\widehat{C}_{pk}^{U(S)} = \frac{b_{m(N-m)}}{c_4} \times \widehat{C}_{pk}^{U(S)}$ is the UMVUE of C_{pk}^{U} for $\mu < T$.

Therefore, combining the two cases, theorem 5 follows.

Similarly,

$$\widehat{C}_{pk}^{L(S)} \sim \begin{cases} \left[\frac{(k+1)(T-L)c_4\sqrt{m(N-m)}}{3k\sigma}\right] \times \chi_{m(N-m)}^{-1} - \frac{c_4}{3k\sqrt{N}} \times t_{m(N-m)}(\delta_L^{(N)}), \text{ for } \mu \geq T \\ \frac{c_4}{3\sqrt{N}} \times t_{m(N-m)}(\delta_L^{(N)}), \quad \text{for} \quad \mu < T \end{cases}$$

Also, $\widehat{C}_{pk}^{L(S)} = \frac{b_{m(N-m)}}{c_4} \times \widehat{C}_{pk}^{L(S)}$ is the UMVUE of C_{pk}^{L}, where, $\widehat{C}_{pk}^{L(S)}$ can be defined analogous to $\widehat{C}_{pk}^{U(S)}$ in equation (28).

Again, similar to the distributions of the plug-in estimators of C_{pm}^{U} based on i) single sample information [see equation (15)] and ii) $\overline{X} - R$ chart information [see equation (24)], the statistical distribution of $\widehat{C}_{pm}^{U(S)}$ can be obtained as

$$\widehat{C}_{pm}^{U(S)} \sim \frac{U-T}{3\sigma} \chi^{*U(S)^{-1}} \tag{31}$$

where, $\widehat{C}_{pm}^{U(S)}$ denotes the plug-in estimator of C_{pm}^{U} based on $\overline{X} - S$ chart information and $\chi^{*U(S)^2} = \frac{1}{m(N-m)\times c_4^2} \times \chi_{m(N-m)}^2 + \frac{I_k^{U^2}}{N} \times \chi_1^2(\delta_1^{(N)^2})$.

Finally, the distribution of the plug-in estimator $\widehat{C}_{pmk}^{U(S)}$ of C_{pmk}^{U} will be

$$\widehat{C}_{pmk}^{U(S)} \sim \frac{U-T}{3\sigma\sqrt{\chi^{*U(S)^2}}} - \frac{1}{3\sqrt{1 + \left(\frac{b_{m(N-m)}}{N}\right) \times \left(\frac{1}{c_4 I_k^U}\right)^2 \times F_{1,m(N-m)}(\delta_1^{(N)^2})}} \tag{32}$$

The distributions of $\widehat{C}_{pm}^{L(S)}$ and $\widehat{C}_{pmk}^{L(S)}$ can be obtained accordingly. Moreover, as in the earlier two situations, here also, the unbiased estimators of $\widehat{C}_{pm}^{I(S)}$ and $\widehat{C}_{pmk}^{I(S)}$ are difficult to obtain.

Thus, for all the three types of estimation procedures discussed in sections 4, 5.1 and 5.2, the newly developed distributions of $\widehat{C}_{pm}^{I}$ and $\widehat{C}_{pmk}^{I}$ are more tractable and consequently, easier to handle as compared to those of Grau (2009). Hence, although, the estimated PCIs do not correspond to any standard statistical distribution; they can, very well, be further utilized, for example, while designing the corresponding process capability control charts. On the contrary, the complicated expressions of the said distributions, obtained through Grau's (2009) approach, do restrict their applications.

6 Process capability Control Charts of $C_p^I(u, v)$

Generally, manufacturing processes tend to have between batch as well as within batch components of variation. Often, capability assessment of a process based on single sample information fail to capture the actual health of the process; whereas, the usual approach of multiple sample estimation of PCIs tend to smooth out some important fluctuations in the capability level of a process through out the entire production cycle. Moreover, even after the stability of a process has been established through suitable control charts, the PCI values measured by samples drawn at specific interval of time are likely to fluctuate from sample to sample due to several reasons [Spiring (1995); Chatterjee and Chakraborty (2013b)]. In fact one major challenge for a production engineer is to decide the time for measuring the capability of a process. Spiring (1995) has argued for using process capability control chart to keep constant vigil on a process.

In this context, stability of a process does not ensure consistent process capability values. This may be due to the fact that, the usual methods of checking stability of a process using suitable control charts, do not take into account the specification limits pertaining to the concerned quality characteristics. Hence, even if a process is found to be stable, the particular quality characteristic value may be highly off-target or may have unacceptable amount of variation with respect the pre-assigned specification limits. Since often, a process is an interface between customer and producer, which usual control chart studies grossly ignore, process capability control charts can be used to assess consistency in the capability values of a process. In fact, it has been observed that unless consistency in the process capability values is established through process capability control charts, the capability of a process should not be summarized based on a single PCI value as those values may be highly subjective and may not even reflect the true capability level of a process [refer

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Chatterjee and Chakraborty (2013b)].

Before computing capability of a process, it is mandatory to check and establish its stability [Kotz and Johnson (2002)] which is generally done by using appropriate control charts. In case of process capability control charts, the information gathered from these control charts are used to estimate the process parameters (viz., μ and σ). Chatterjee and Chakraborty (2013b) have designed process capability control charts of C_{PU} and C_{PL} based on information from $\overline{X}-R$ and $\overline{X}-S$ charts. However, as has been discussed in section 1, C_{PU} and C_{PL} have a number of drawbacks. Hence, for quality characteristics with unilateral specification limits, control charts for $C_p^I(u,v)$, with $u = 0, 1$ and $v = 0, 1$ need to be developed. Here, we construct the control charts of $C_p^U(u,v)$ while those for $C_p^L(u,v)$ can be developed similarly.

6.1 Process Capability Control Charts of C_p^U

Case I: Based on Information from $\overline{X} - R$ charts

From equation (19),

$$P\left[b_\nu\sqrt{\nu}\frac{U-T}{3\sigma}\chi^{-1}_{1-\alpha/2,\nu} \leq \widetilde{C}_p^{U(R)} \leq b_\nu\sqrt{\nu}\frac{U-T}{3\sigma}\chi^{-1}_{\alpha/2,\nu}\right] = 1-\alpha$$

Also, the control limits developed directly from the statistical distribution of C_p^U [see equation (19)] involve μ and σ^2 and hence are often unobservable. To address this problem, those parameters should be replaced by their estimators obtained from the corresponding $\overline{X} - R$ and $\overline{X} - S$ charts, which ever is applicable. Thus, the control limits of the process capability control chart of C_p^U, based on the information from the corresponding $\overline{X} - R$ charts, are given by

$$\left.\begin{aligned} UCL^{(R)}_{C_p^U} &= b_\nu\sqrt{\nu}\times\overline{\widetilde{C}}_p^{U(*R)}\chi^{-1}_{\alpha/2,\nu} \\ CL^{(R)}_{C_p^U} &= \overline{\widetilde{C}}_p^{U(*R)} \\ LCL^{(R)}_{C_p^U} &= b_\nu\sqrt{\nu}\times\overline{\widetilde{C}}_p^{U(*R)}\chi^{-1}_{1-\alpha/2,\nu} \end{aligned}\right\} \quad (33)$$

Here, $\overline{\widetilde{C}}_p^{U(*R)} = \frac{1}{m}\sum_{i=1}^{m}\widetilde{C}_{p_i}^{U(*R)}$ and $\widetilde{C}_{p_i}^{U(*R)}$ is an unbiased estimator of C_p^U based on individual sample information of $\overline{X}-R$ chart. It is easy to see that for any individual subgroup, $\widetilde{C}_p^{U(*R)} = \frac{cb_{\nu_1}}{d_2}\widehat{C}_p^{U(*R)}$ with $\widehat{C}_p^{U(*R)} = \frac{d_2(U-T)}{3R}$ is the corresponding plug-in estimator of C_p^U.

Here, following Kuo (2010), ν_1 and c can be defined as $\nu_1 = \frac{1}{-2+2\sqrt{1+\frac{2d_3^2}{d_2^2}}}$ and $c = d_2 \times \sqrt{\frac{\nu_1}{2}}\frac{\Gamma(\frac{\nu_1}{2})}{\Gamma(\frac{\nu_1+1}{2})}$. Note that, here ν_1 is defined by substituting $m = 1$ in the definition of ν for individual subgroups. In this context, while defining plug-in estimators of the respective PCIs for individual subgroups, μ and σ are replaced by $\overline{X}$ and $\frac{R}{d_2}$ respectively, instead of using $\overline{\overline{X}}$ and $\frac{\overline{R}}{d_2}$.

Case II: Based on Information from $\overline{X}-S$ charts

From equation (26),

$$P\left[\frac{b_{m(N-m)}(U-T)\sqrt{m(N-m)}}{3\sigma}\times\chi^{-1}_{1-\alpha/2,m(N-m)} \le \widetilde{C}_p^{U(S)} \le \frac{b_{m(N-m)}(U-T)\sqrt{m(N-m)}}{3\sigma}\times\chi^{-1}_{\alpha/2,m(N-m)}\right] = 1-\alpha$$

Then, the control limits of the C_p^U control chart, based on information from the corresponding $\overline{X}-S$ chart, are given by

$$\left.\begin{aligned} UCL^{(S)}_{C_p^U} &= b_{m(N-m)}\sqrt{m(N-m)}\times\overline{\widetilde{C}}_p^{U(*S)}\chi^{-1}_{\alpha/2,m(N-m)}\\ CL^{(S)}_{C_p^U} &= \overline{\widetilde{C}}_p^{U(*S)}\\ LCL^{(S)}_{C_p^U} &= b_{m(N-m)}\sqrt{m(N-m)}\times\overline{\widetilde{C}}_p^{U(*S)}\chi^{-1}_{1-\alpha/2,m(N-m)}\end{aligned}\right\} \tag{34}$$

Here, $\overline{\widetilde{C}}_p^{U(*S)} = \frac{1}{m}\sum_{i=1}^{m}\widetilde{C}_{p_i}^{U(*S)}$ and $\widetilde{C}_{p_i}^{U(*S)}$ is the UMVUE of C_p^U based on individual sample information of $\overline{X}-S$ chart for the i^{th} subgroup, for $i = 1(1)m$. For any individual subgroup, $\widetilde{C}_p^{U(*S)} = b_{n-1}\widehat{C}_p^{U(*S)}$ with $\widehat{C}_p^{U(*S)} = \frac{U-T}{3S}$ being the corresponding plug-in estimator of C_p^U. Here, while defining plug-in estimators of the respective PCIs for individual subgroups, μ and σ are replaced by $\overline{X}$ and S respectively.

6.2 Process Capability Control Charts of C_{pk}^U

Case I: Based on Information from $\overline{X}-R$ charts

When $\mu \geq T$: Here, $\widehat{C}_{pk}^{U(R)} = \widehat{C}_{PU}^{(R)}$ and hence the control limits of C_{pk}^{U} control chart based on $\overline{X} - R$ chart information will be the same as those of the corresponding C_{PU} control chart as developed by Chatterjee and Chakraborty (2013b).

When $\mu < T$: Here, from equation (23),

$$P\left[\left(\frac{(k+1)(U-T)b_\nu\sqrt{\nu}}{3k\sigma}\right)\times\chi^{-1}_{1-\alpha/2,\nu} - \frac{b_\nu}{3k\sqrt{N}}\times t_{1-\alpha/2,\nu}(\delta_U^{(N)}) \leq \widehat{C}_{pk}^{U(R)} \leq \left(\frac{(k+1)(U-T)b_\nu\sqrt{\nu}}{3k\sigma}\right)\times\chi^{-1}_{\alpha/2,\nu} - \frac{b_\nu}{3k\sqrt{N}}\times t_{\alpha/2,\nu}(\delta_U^{(N)})\right] = 1-\alpha$$

Thus, the control limits of C_{pk}^{U} control chart, based on information from the corresponding $\overline{X} - R$ charts, will be as follows:

$$\left.\begin{aligned}
UCL_{C_{pk}^U}^{(R)} &= \left(\frac{(k+1)(U-T)b_\nu\sqrt{\nu}}{[k(U-T)-(T-\mu)]}\right)\times\overline{\widetilde{C}}_{pk}^{U(*R)}\chi^{-1}_{\alpha/2,\nu} - \frac{b_\nu}{3k\sqrt{N}}\times t_{\alpha/2,\nu}(\overline{\widetilde{\delta}}_U^{(N,R)})\\
CL_{C_{pk}^U}^{(R)} &= \overline{\widetilde{C}}_{pk}^{U(*R)}\\
LCL_{C_{pk}^U}^{(R)} &= \left(\frac{(k+1)(U-T)b_\nu\sqrt{\nu}}{[k(U-T)-(T-\mu)]}\right)\times\overline{\widetilde{C}}_{pk}^{U(*R)}\chi^{-1}_{1-\alpha/2,\nu} - \frac{b_\nu}{3k\sqrt{N}}\times t_{1-\alpha/2,\nu}(\overline{\widetilde{\delta}}_U^{(N,R)})
\end{aligned}\right\} \quad (35)$$

Here, $\overline{\widetilde{C}}_{pk}^{U(*R)} = \frac{1}{m}\sum_{i=1}^{m}\widetilde{C}_{pk_i}^{U(*R)}$ and $\widetilde{C}_{pk_i}^{U(*R)}$ is an unbiased estimator of C_{pk}^{U} based on individual sample information of $\overline{X} - R$ chart, such that, for any individual subgroup, $\widetilde{C}_{pk}^{U(*R)} = \frac{cb_{\nu_1}}{d_2}\times\widehat{C}_{pk}^{U(*R)}$, where, $\widehat{C}_{pk}^{U(*R)}$ is the corresponding plug-in estimator of C_{pk}^{U}. Also, $\overline{\widetilde{\delta}}_U^{(N,R)} = 3\sqrt{N}\,\overline{\widetilde{C}}_p^{U(*R)}$.

Case II: Based on Information from $\overline{X} - S$ charts

When $\mu \geq T$: Here, $\widehat{C}_{pk}^{U(S)} = \widehat{C}_{PU}^{(S)}$ and hence here also the control limits of C_{pk}^{U} control chart based on $\overline{X} - S$ chart information will be the same as those of the corresponding C_{PU} control chart as developed by Chatterjee and Chakraborty (2013b).

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When $\mu < T$: Here, from equation (30),

$$P\left[\left(\frac{(k+1)(U-T)b_{m(N-m)}\sqrt{m(N-m)}}{3k\sigma}\right)\times\chi^{-1}_{1-\alpha/2,m(N-m)} - \frac{b_{m(N-m)}}{3k\sqrt{N}}\times t_{1-\alpha/2,m(N-m)}(\delta_U^{(N)}) \le \widetilde{C}_{pk}^{U(S)} \le \left(\frac{(k+1)(U-T)b_{m(N-m)}\sqrt{m(N-m)}}{3k\sigma}\right)\times\chi^{-1}_{\alpha/2,m(N-m)} - \frac{b_{m(N-m)}}{3k\sqrt{N}}\times t_{\alpha/2,m(N-m)}(\delta_U^{(N)})\right] = 1-\alpha$$

Thus, the control limits of C_{pk}^U control chart, based on information from the corresponding $\overline{X}-S$ charts, will be as follows:

$$\left.\begin{aligned} UCL^{(S)}_{C^U_{pk}} &= \left(\frac{(k+1)(U-T)b_{m(N-m)}\sqrt{m(N-m)}}{k(U-T)-(T-\mu)}\right)\times\overline{\widetilde{C}}_{pk}^{U(*S)}\times\chi^{-1}_{\alpha/2,m(N-m)} - \frac{b_{m(N-m)}}{3k\sqrt{N}}\times t_{1-\alpha/2,m(N-m)}(\overline{\widetilde{\delta}}_U^{(N,S)}) \\ CL^{(S)}_{C^U_{pk}} &= \overline{\widetilde{C}}_{pk}^{U(*S)} \\ LCL^{(S)}_{C^U_{pk}} &= \left(\frac{(k+1)(U-T)b_{m(N-m)}\sqrt{m(N-m)}}{k(U-T)-(T-\mu)}\right)\times\overline{\widetilde{C}}_{pk}^{U(*S)}\times\chi^{-1}_{1-\alpha/2,m(N-m)} - \frac{b_{m(N-m)}}{3k\sqrt{N}}\times t_{1-\alpha/2,m(N-m)}(\overline{\widetilde{\delta}}_U^{(N,S)}) \end{aligned}\right\} \quad (36)$$

Here, $\overline{\widetilde{C}}_{pk}^{U(*S)} = \frac{1}{m}\sum_{i=1}^{m}\widetilde{C}_{pk_i}^{U(*S)}$ and $\widetilde{C}_{pk_i}^{U(*S)}$ is the UMVUE of C_{pk}^U based on individual sample information of $\overline{X}-S$ chart. For any individual subgroup, $\widetilde{C}_{pk}^{U(*S)} = b_{n-1}\,\widehat{C}_{pk}^{U(*S)}$, where, $\widehat{C}_{pk}^{U(*S)}$ is the corresponding plug-in estimator of C_{pk}^U. Also, $\overline{\widetilde{\delta}}_U^{(N,S)} = 3\sqrt{N}\,\overline{\widetilde{C}}_{pk}^{U(*S)}$.

6.3 Process Capability Control Charts of C_{pm}^U

Case I: Based on Information from $\overline{X}-R$ charts

From equation (24), the control limits of C_{pm}^U control chart, based on information from the corresponding $\overline{X}-R$ charts will be,

$$\left.\begin{aligned} UCL^{(R)}_{C^U_{pm}} &= \sqrt{1+\overline{\widehat{\delta}}_U^{*(R)^2}}\times\overline{\widehat{C}}_{pm}^{U(R)}\times\left[\overline{\widehat{\chi}}_{\alpha/2}^{*U(R)}\right]^{-1} \\ CL^{(R)}_{C^U_{pm}} &= \overline{\widehat{C}}_{pm}^{U(R)} \\ LCL^{(R)}_{C^U_{pm}} &= \sqrt{1+\overline{\widehat{\delta}}_U^{*(R)^2}}\times\overline{\widehat{C}}_{pm}^{U(R)}\times\left[\overline{\widehat{\chi}}_{1-\alpha/2}^{*U(R)}\right]^{-1} \end{aligned}\right\} \quad (37)$$

Here, $\overline{\widehat{C}}_{pm}^{U(R)} = \frac{1}{m}\sum_{i=1}^{m}\widehat{C}_{pm_i}^{U(*R)}$ and $\widehat{C}_{pm_i}^{U(*R)}$ is the plug-in estimator of C_{pm}^U based on individual sample information of $\overline{X}-R$ chart. Also, $\overline{\widehat{\chi}}^{*U(R)} = \frac{1}{\nu}\left(1+\frac{d_3^2}{md_2^2}\right)\chi^2_\nu + \frac{I_k^{U2}}{N}\chi^2_1(\overline{\widehat{\delta}}_1^{(N,R)})$, where, $\overline{\widehat{\delta}}_U^{*(R)}$ and $\overline{\widehat{\delta}}_1^{(N,S)}$

are defined accordingly.

Case II: Based on Information from $\overline{X}-S$ charts

From equation (31), the control limits of the C_{pm}^{U} control chart, based on $\overline{X}-S$ chart information, will be,

$$\left.\begin{aligned} UCL_{C_{pm}^{U}}^{(S)} &= \sqrt{1+\overline{\widehat{\delta}}_{U}^{*(S)^2}} \times \overline{\widehat{C}}_{pm}^{U(S)} \times \left[\overline{\widehat{\chi}}_{\alpha/2}^{*U(S)}\right]^{-1} \\ CL_{C_{pm}^{U}}^{(S)} &= \overline{\widehat{C}}_{pm}^{U(S)} \\ LCL_{C_{pm}^{U}}^{(S)} &= \sqrt{1+\overline{\widehat{\delta}}_{U}^{*(S)^2}} \times \overline{\widehat{C}}_{pm}^{U(S)} \times \left[\overline{\widehat{\chi}}_{1-\alpha/2}^{*U(S)}\right]^{-1} \end{aligned}\right\} \quad (38)$$

Here, $\overline{\widehat{C}}_{pm}^{U(S)} = \frac{1}{m}\sum_{i=1}^{m}\widehat{C}_{pm_i}^{U(*S)}$ and $\widehat{C}_{pm_i}^{U(*S)}$ is the plug-in estimator of C_{pm}^{U} based on individual sample information of $\overline{X}-S$ charts. Also, $\overline{\widehat{\chi}}^{*U(S)} = \frac{1}{mNC_4^2}\chi_{m(N-m)}^2 + \frac{I_k^{U2}}{N}\chi_1^2(\overline{\widehat{\delta}}_1^{(N,S)})$, where, $\overline{\widehat{\delta}}_{U}^{*(S)^2}$ and $\overline{\widehat{\delta}}_1^{(N,S)}$ are defined accordingly.

6.4 Process Capability Control Charts of C_{pmk}^{U}

Case I: Based on Information from $\overline{X}-R$ charts

From equation (25), the control limits of the process capability control chart of C_{pmk}^{U} will be

$$\left.\begin{aligned} UCL_{C_{pmk}^{U}}^{(R)} &= \frac{\sqrt{1+\overline{\widehat{\delta}}_{U}^{*(R)^2}} \times \overline{\widehat{C}}_{pmk}^{U(R)}}{(1-\overline{\widehat{\alpha}}_{U})\sqrt{\overline{\widehat{\chi}}_{\alpha/2}^{*U(R)^2}}} - \left[3\sqrt{1+N\times\left(\frac{d_2^*}{d_2 I_k^U}\right)^2 \times F_{\alpha/2,1,\nu}\left(\overline{\widehat{\delta}}_1^{(N,R)^2}\right)}\right]^{-1} \\ CL_{C_{pmk}^{U}}^{(R)} &= \overline{\widehat{C}}_{pmk}^{U(R)} \\ LCL_{C_{pmk}^{U}}^{(R)} &= \frac{\sqrt{1+\overline{\widehat{\delta}}_{U}^{*(R)^2}} \times \overline{\widehat{C}}_{pmk}^{U(R)}}{(1-\overline{\widehat{\alpha}}_{U})\sqrt{\overline{\widehat{\chi}}_{\alpha/2}^{*U(R)^2}}} - \left[3\sqrt{1+N\times\left(\frac{d_2^*}{d_2 I_k^U}\right)^2 \times F_{\alpha/2,1,\nu}\left(\overline{\widehat{\delta}}_1^{(N,R)^2}\right)}\right]^{-1} \end{aligned}\right\} \quad (39)$$

Here, $\overline{\widehat{C}}_{pmk}^{U(R)} = \frac{1}{m}\sum_{i=1}^{m}\widehat{C}_{pmk_i}^{U(*R)}$ and $\widehat{C}_{pmk_i}^{U(*R)}$ is the plug-in estimator of C_{pmk}^{U} based on individual sample information of $\overline{X}-R$ chart.

Case II: Based on Information from $\overline{X} - S$ charts

Based on equation (32), the control limits of C_{pmk}^{U} control chart, with information from the corresponding $\overline{X} - S$ charts, can be obtained as follows:

$$\left.\begin{array}{l} UCL_{\widehat{C}_{pmk}^{U(S)}} = \dfrac{\sqrt{1+\overline{\widehat{\delta}}_{U}^{*(S)^2}} \times \overline{\widehat{C}}_{pmk}^{U(S)}}{(1-\overline{\overline{\widehat{\alpha}}}_{U})\sqrt{\overline{\widehat{\chi}}_{\alpha/2}^{*U(S)^2}}} - \left[3\sqrt{1+\dfrac{b_{m(N-m)}}{N\left(c_4 I_k^U\right)^2} \times F_{\alpha/2,1,m(N-m)}\left(\overline{\widehat{\delta}}_{1}^{(N,S)^2}\right)}\,\right]^{-1} \\ CL_{\widehat{C}_{pmk}^{U(S)}} = \overline{\widehat{C}}_{pmk}^{U(S)} \\ LCL_{\widehat{C}_{pmk}^{U(S)}} = \dfrac{\sqrt{1+\overline{\widehat{\delta}}_{U}^{*(S)^2}} \times \overline{\widehat{C}}_{pmk}^{U(S)}}{(1-\overline{\overline{\widehat{\alpha}}}_{U})\sqrt{\overline{\widehat{\chi}}_{1-\alpha/2}^{*U(S)^2}}} - \left[3\sqrt{1+\dfrac{b_{m(N-m)}}{N\left(c_4 I_k^U\right)^2} \times F_{1-\alpha/2,1,m(N-m)}\left(\overline{\widehat{\delta}}_{1}^{(N,S)^2}\right)}\,\right]^{-1} \end{array}\right\} \quad (40)$$

Here, $\overline{\widehat{C}}_{pmk}^{U(S)} = \frac{1}{m}\sum_{i=1}^{m}\widehat{C}_{pmk_i}^{U(*S)}$ where, $\widehat{C}_{pmk_i}^{U(*S)}$ is the plug-in estimator of C_{pmk}^{U} based on individual sample information from $\overline{X} - S$ charts.

7 Numerical Examples

In order to discuss practical application of the theory developed so far in the present article, we now consider two numerical examples.

7.1 Example 1

We first consider the data set originally used by Chatterjee and Chakraborty (2013b). This data pertains to a chemical industry. The quality characteristic, which is of the smaller the better type, is coded as 'X'. The USL and the target for this quality characteristic are set as $U = 0.3$ unit and $T = 0.16$ unit respectively. Also, from the said data set, the summary statistics are found to be as $m = 6,\ n = 5,\ \overline{\overline{X}} = 0.1577 < T$ and $\overline{R} = 0.055$.

Suppose, loss of profit for per 0.01 unit deviation from T towards left is \$0.05 and constant selling price per item is \$5. Then, using the formulation given by Chatterjee and Chakraborty (2012), we have, $k = 4.138$.

Now, before computing various PCI values based on this data set, we need to construct the corresponding process capability control charts for investigating the consistency in the capability level of the process over various subgroups of samples. Since here the sample size 'n' is considerably small ($n < 10$), the control limits of the process capability control charts for various processes should be based on the information from the corresponding $\overline{X} - R$ control charts.

The control limits of the process capability control charts of C_p^U, C_{pk}^U, C_{pm}^U and C_{pmk}^U are given below with the corresponding control charts shown in Figures 1 − 4 respectively.

$$\left.\begin{array}{l} UCL^{(R)}_{C_p^L} = 2.3082 \\ CL^{(R)}_{C_p^L} = 1.6892 \\ LCL^{(R)}_{C_p^L} = 1.2631 \end{array}\right\} \quad (41)$$

$$\left.\begin{array}{l} UCL^{(R)}_{C_{pk}^L} = 2.2972 \\ CL^{(R)}_{C_{pk}^L} = 1.6619 \\ LCL^{(R)}_{C_{pk}^L} = 1.2602 \end{array}\right\} \quad (42)$$

$$\left.\begin{array}{l} UCL^{(R)}_{C_{pm}^L} = 2.7090 \\ CL^{(R)}_{C_{pk}^L} = 1.9715 \\ LCL^{(R)}_{C_{pk}^L} = 1.4790 \end{array}\right\} \quad (43)$$

$$\left.\begin{array}{l} UCL^{(R)}_{C_{pmk}^L} = 2.6333 \\ CL^{(R)}_{C_{pmk}^L} = 1.9401 \\ LCL^{(R)}_{C_{pmk}^L} = 1.4788 \end{array}\right\} \quad (44)$$

FIGURES 1 – 4 **SHOULD BE ABOUT HERE**

Since all the PCI values corresponding to all the PCI control charts, given in Figures 1 − 4, lie within the respective control limits, it is logical to expect that the process is consistently capable of performing satisfactorily. Note that, the control charts for C_p^U and C_{pk}^U are based on their unbiased estimator values for individual subgroups, while, the charts for C_{pm}^U and C_{pmk}^U are based on the corresponding plug-in estimators only (due the unavailability of their unbiased estimators).

Based on the given data, $\widetilde{C}_p^{U(R)} = 1.9736$, $\widehat{C}_p^{U(R)} = 1.9242$, $\widetilde{C}_{pk}^{U(R)} = 1.9657$, $\widehat{C}_{pk}^{U(R)} = 1.9166$, $\widehat{C}_{pm}^{U(R)} = 1.9730$ and $\widehat{C}_{pmk}^{U(R)} = 1.9652$.

It is easy to check that for individual subgroups, the values of the plug-in estimators for all these four PCIs follow the said interrelationship. Also, since all the estimated PCI values are within the corresponding UCL and LCL, irrespective of the choice of the PCI, it is logical to expect that

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the process is having consistent capability over the production cycle and hence the situation is favourable for overall assessment of the process through a single PCI value.

Using $k = 4.138$ and from equation(10), the threshold value of C_p^U will be $C_p^{U(T)} = 0.389$. Thus, the process can be considered to be performing satisfactorily. Also, since $\overline{\overline{X}} < T$, $\widehat{P}_{NC}^{U(E)} = 2.254 \times 10^{-9}$, i.e. $NCPPM^E = 2.254 \times 10^{-6}$ which is quite small and hence justifies the high values of the PCIs. On the other hand, $\widehat{P}_{NC}^{U} = 3.903 \times 10^{-9}$ i.e., $NCPPM = 3.903 \times 10^{-6}$. Thus, similar to the case of asymmetric specification limits, here also, P_{NC}^U does not always give minimum observable PNC. This is due to that fact that here, $\overline{\overline{X}} < T$ and thus, the average quality level is actually better than that of the so called "potential quality level" i.e. $\mu = T$.

Note that, although, in the present example, the process is found to be stable as well as consistently capable, this may not always be the case. In the following example, we shall discuss about such a process.

7.2 Example 2

While discussing about the process capability control charts of C_{PU} and C_{PL}, Chen et al. (2007) have considered a dataset from a integrated circuit (IC) manufacturing process, where one of the major quality characteristics is wire bonding of gold wire. This is a quality characteristic of higher the better type with lower specification limit (LSL) being 5 mm. The dataset consists of 25 sub-groups each having 11 sample observations. Thus, $L = 5$, $m = 25$ and $n = 11$. Since, here the sample size is considerably large, data gathered from the corresponding $\overline{X} - S$ chart are used to construct the required process capability control charts. These charts are given in Figures 5 and 6 below:

FIGURES 5 and 6 SHOULD BE ABOUT HERE

From Figures 5 and 6, it is evident that the process is stable and hence we can proceed to assess its capability. Now, since in the original dataset provided by Chen et al. (2007), sufficient information, required for the computation of 'k', is not available, let us consider 'k' to have the same value as in example 1, i.e., $k = 4.138$.

Then, the control limits of the process capability control charts for C_p^L, C_{pk}^L, C_{pm}^L and C_{pmk}^L can be obtained respectively as follows:

$$\left.\begin{aligned} UCL_{C_p^L}^{(S)} &= 1.3551 \\ CL_{C_p^L}^{(S)} &= 1.3313 \\ LCL_{C_p^L}^{(S)} &= 1.3084 \end{aligned}\right\} \quad (45)$$

$$\left.\begin{aligned} UCL_{C_{pk}^L}^{(S)} &= 1.6083 \\ CL_{C_{pk}^L}^{(S)} &= 1.2720 \\ LCL_{C_{pk}^L}^{(S)} &= 1.2564 \end{aligned}\right\} \quad (46)$$

$$\left.\begin{aligned} UCL_{C_{pm}^L}^{(S)} &= 1.4111 \\ CL_{C_{pk}^L}^{(S)} &= 1.3988 \\ LCL_{C_{pk}^L}^{(S)} &= 1.3533 \end{aligned}\right\} \quad (47)$$

$$\left.\begin{aligned} UCL_{C_{pmk}^L}^{(S)} &= 1.4144 \\ CL_{C_{pmk}^L}^{(S)} &= 1.3386 \\ LCL_{C_{pmk}^L}^{(S)} &= 1.3111 \end{aligned}\right\} \quad (48)$$

Also, the corresponding control charts are given in Figures 7 – 10.

FIGURES 7 – 10 **SHOULD BE ABOUT HERE**

Figures 5−10 reveal an interesting fact. Unlike the process capability control charts for example 1, here, despite being stable (see Figures 5 and 6), the process fails to prove itself as consistently capable (see Figures 7 – 10) as around 15 out of the 25 subgroup PCIs lie below the LCL for all the four PCIs viz., C_p^L, C_{pk}^L, C_{pm}^L and C_{pmk}^L. Note that, since PCIs are generally of higher the better type, only subgroups having PCI values less than the respective LCLs of the process capability control charts are of the concern. In fact, for a process capability control chart, LCL signifies that under the prevailing process centering as well as process dispersion scenario, the concerned quality characteristic should be able to achieve atleast LCL amount of capability value; while subgroups with PCI values higher than UCL are of satisfactory quality. However, for all the four PCIs, the index values corresponding to subgroups 24 and 25 need more exploration as those are grossly deviated from the UCL. This may indicate certain change in centering and/or dispersion level of the concerned quality characteristic. An inner-view into the process indeed reveals that the level of variation is considerably smaller for these two subgroups as compared to the remaining subgroups. Chen et al.'s (2007) control chart for $C)PU$ and C_{PL} failed to capture this aspect of the process. Thus, since the process capability values of the process are highly unstable, the usual single valued capability assessment of the process is not solicited at this stage of production [refer Chatterjee and Chakraborty (2013b)]. For example, the values of the plug-in estimators of C_p^L, C_{pk}^L, C_{pm}^L and C_{pmk}^L, based on the information from the corresponding $\overline{X}-S$ chart are $\widehat{C}_p^{L(s)} = 1.328441$, $\widehat{C}_{pk}^{L(s)} = 1.3278$, $\widehat{C}_{pm}^{L(s)} = 1.328396$ and $\widehat{C}_{pmk}^{L(s)} = 1.3256$. Also, the UMVUEs of C_p^L and C_{pk}^L are $\widetilde{C}_p^{L(s)} = 1.3619$ and $\widetilde{C}_{pk}^{L(s)} = 1.3613$. It is easy to observe that these sample PCI values tend to average out the actual fluctuations in the subgroup level PCI values. For example, UMVUEs of C_{pk}^L values at subgroup level, range from 0.8339 (corresponding to subgroup 3) to 2.2633 (corresponding to subgroup 24); which is almost averaged out through $\widetilde{C}_{pk}^{L(s)} = 1.3613$. Such situation is valid for the other three PCIs as well.

Another interesting point to note from both of these examples is that, unlike example 1, where, the control limits are wide apart - to accommodate all the subgroup PCIs; for example 2, the control limits are very closely aligned. This is due to the fact that the control limits formulated in the present article are functions of the number of subgroups (m) and the sample size (n) and they come closer if atleast one of 'm' and 'n' gets increased. This property can also be observed in the cases of the so called $\overline{X}$, S and R charts and also the process capability control charts for C_{PU} and C_{PL} designed by Chatterjee and Chakraborty (2013b).

8 Conclusions

$C_p^U(u,v)$ and $C_p^L(u,v)$ [jointly expressed as $C_p^I(u,v)$, vide equation (2)], are two very important classes of PCIs, developed by Grau (2009), for quality characteristics having unilateral specification limits. Along with the expression of 'k' [vide equation (3)], as suggested by Chatterjee and

Chakraborty (2012), $C_p^I(u, v)$ addresses almost all the drawbacks of the existing PCIs for unilateral specification limits, from both the distributional and interpretational viewpoint.

In the present article, we have discussed about some important statistical properties of $C_p^U(u, v)$ and $C_p^L(u, v)$. We have developed relationship between C_p^I, C_{pk}^I and proportion of non-conformance, for the situation where, $\mu \neq T$ and have observed that unlike the symmetric bilateral specification limits, here the production of items on target do not always ensure production of minimum attainable PNC. We have also formulated the expressions for the threshold values C_p^I. Contradicting the usual convention of considering '1' as the threshold value of any PCI irrespective of the nature of the specification limits; our expression for the threshold value show that for unilateral specification limits, the threshold value is not unique and is always smaller than '1'. $C_p^I(u, v)$ being a comparatively new super-structure of PCIs as compared to the other existing PCIs for unilateral specification limits, these crucial statistical properties of it were hardly explored before in literature.

Next, we have studied the distributional properties of the member indices of $C_p^I(u, v)$. Although Grau (2009) had already studied these properties earlier, his expressions involve difficult mathematical formulation and hence are unsuitable for further application. On the contrary, we have formulated more tractable distributions of these PCIs based on single sample information as well as information gathered from $\overline{X} - R$ and $\overline{X} - S$ charts. Moreover, for C_p^I and C_{pk}^I, the corresponding UMVUEs (or unbiased estimators, when $\overline{X} - R$ chart information is used) have been developed. We have also designed the process capability control charts of $C_p^I(u, v)$ for $u = 0, 1$ and $v = 0, 1$.

Finally, we have discussed two numerical examples to validate our theoretical findings discussed in this article. It has been observed that, although stability is a necessary condition to be satisfied before computing PCI values; it is not the sufficient. Particularly, the process described in example 2 is stable but does not have consistent capability. In fact, due to such unwarranted fluctuations in PCI values over the subgroups, summarization of the overall process capability through the use of a single PCI value is not solicited. There is no denying of the fact that proper interpretation as well as apt application of a PCI is a key to successful implementation of the process capability studies in a process. The present article, grossly, puts emphasis on this very fact by studying various crucial distributional and interpretational aspects of some PCIs for unilateral specification limits and by discussing some prospective areas of applications.

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