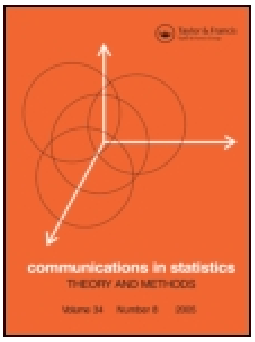

Communications in Statistics - Theory and Methods

ISSN: 0361-0926 (Print) 1532-415X (Online) Journal homepage: http://www.tandfonline.com/loi/lsta20

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To cite this article: Moutushi Chatterjee & Ashis Kumar Chakraborty (2015) Distributions and Process Capability Control Charts for C_{PU} and C_{PL} Using Subgroup Information, Communications in Statistics - Theory and Methods, 44:20, 4333-4353, DOI: 10.1080/03610926.2013.851233

To link to this article: http://dx.doi.org/10.1080/03610926.2013.851233

Published online: 07 Oct 2015.

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Communications in Statistics—Theory and Methods, 44: 4333–4353, 2015
Copyright © Taylor & Francis Group, LLC
ISSN: 0361-0926 print / 1532-415X online
DOI: 10.1080/03610926.2013.851233

Distributions and Process Capability Control Charts for C_{PU} and C_{PL} Using Subgroup Information

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Process capability analysis is a very well-known and widely accepted method of assessing the ability of a process to produce items within pre-assigned specification limits. Most of the process capability indices (PCI) available in literature are formulated in terms of the parameters of the concerned quality characteristics. However, since the actual values of these parameters are often unknown, their estimated values are used to evaluate the estimated capability of a process. One such estimation procedure may be to use the estimates of these parameters obtained from the corresponding control charts used to check the stability of the said process. In this article, we used this approach to redefine plug-in (natural) estimators of the two most famous PCI's for unilateral specification limits viz., C_{PU} and C_{PL}. We formulated the corresponding unbiased estimators and uniformly minimum variance unbiased estimators (UMVUE), wherever possible, and their distributions as well. We also designed the process capability control charts of C_{PU} and C_{PL} based on these UMVUEs. For constructing these control charts, we used the estimators of the parameters of the quality characteristics as obtained from the corresponding $\overline{X} - S$ and $\overline{X} - R$ charts. These charts can be used to check the consistency of capability of a process and also to keep a constant vigil on the process. Two numerical examples have been discussed and it has been observed that our proposed process capability control charts are more efficient to detect changes in process capability than those already available in literature.

Keywords Control chart; Process capability index; Subgroup estimate; Uniformly minimum unbiased estimator (UMVUE); Unbiased estimator; Unilateral specification limits.

Mathematics Subject Clssification 62E15; 62F10; 62F25.

1. Introduction

In today's highly competitive market, importance of the correct assessment of the capability of a process to produce items within the pre-assigned specification limits is increasing day by day. Process capability analysis is the most used statistical quality control (SQC) methodology which is applied to carry out such assessment, especially in manufacturing

Received May 7, 2013; Accepted September 26, 2013.
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processes. Such analysis is generally done through process capability indices (PCI). A PCI measures the ability of a process to produce what it is supposed to produce.

Suppose that U and L denote the upper specification limit (USL) and lower specification limit (LSL), respectively, of a quality characteristic. Although most of the PCI's available in literature are designed for processes with bi-lateral specification limits, i.e., processes where the concerned quality characteristic has a USL and a LSL, it is often seen that the design intent of a manufactured product provides information on either of USL or LSL. This is true for quality characteristics of either smaller the better type (e.g., surface roughness, flatness, degree of radiation, tool wear, etc.) or larger the better type (tensile strength, compressive strength, etc.). This necessitates the use of PCIs especially designed for processes with unilateral specifications. Most of the research works on PCIs for unilateral specification limits available in literature are based on C_{PU} and C_{PL} which are defined as follows:

$$\left.\begin{aligned} C_{PU} &= \frac{U-\mu}{3\sigma} \\ \text{and} \quad C_{PL} &= \frac{\mu-L}{3\sigma} \end{aligned}\right\}. \tag{1}$$

Chatterjee and Chakraborty (2012) made an extensive review of the PCIs for unilateral specification limits.

Often, PCIs are expressed in terms of the corresponding parameters (e.g., μ, σ) of the concerned quality characteristics. Since actual values of these parameters are mostly unknown, the values of the PCIs are also often unobservable and this necessitates the estimation of the PCI values for all practical purposes. In a majority of cases, capability of a process is measured by taking a single sample at some point of time. Thus, let "X" denote the quality characteristic under consideration such that $X \sim N(\mu, \sigma^2)$. Also, suppose, a sample of size "n" is taken from the process. Then, $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ denote the sample average and sample variance respectively. Usually, the plug-in estimators of the PCIs, are defined by replacing μ and σ^2 by $\overline{X}$ and S^2, respectively. Thus, from Eq. (1), the plug-in estimators of C_{PU} and C_{PL} will be $\widehat{C}_{PU} = \frac{U-\overline{X}}{3S}$ and $\widehat{C}_{PL} = \frac{\overline{X}-L}{3S}$ and their properties have been studied by Pearn and Chen (2002). Later, Shu et al. (2006) redefined the plug-in estimators of C_{PU} and C_{PL} for the cases where, μ and σ are replaced by the information obtained from multiple samples of various sizes and have studied the properties of those estimators as well.

Before computing capability value of a process, one needs to establish stability of the process using suitable control charts as the PCI value of an unstable process is hardly reliable (Kotz and Johnson, 1993, 2002). Spiring (1995) argued for defining the plug-in estimators of PCIs based on the estimators of the parameters of the concerned quality characteristics as obtained from the respective control charts. Since these estimators are based on the information of a stable process, they are likely to be more robust with respect to sampling fluctuation. Spiring (1995) defined such estimators for $C_{pm} = \frac{d}{3\sqrt{\sigma^2+(\mu-T)^2}}$— a well-known PCI for bi-lateral specification limits. In this article, we used this approach to re-define plug-in estimators of C_{PU} and C_{PL}. We also studied the underlying statistical distributions of these estimators and have found their unbiased estimators and UMVUEs (where ever applicable).

However, the usual approach of such "one shot estimation" of the capability of a process suffers from a number of drawbacks which are listed below.

1. A process, as the name suggests, is an on-going activity and hence the PCI value at a single point of time may not necessarily represent the overall capability scenario of a process (see Spiring, 1995). Even the usual estimation procedure based on multiple

samples (see Shu et al., 2006) is not appropriate as that may smooth-out some important fluctuations in the process and as a result, some serious problems in the process may be overlooked. In fact, one challenging problem for the production engineer is to identify when to measure the capability of a process.
2. Spiring (1995) has also observed that "one shot" estimation of the capability of a process requires considerably large sample size (at least 50).
3. Once the stability of a process is established, estimates of the parameters of the quality characteristics can be obtained from the same control charts. However, these estimates are never used in the subsequent stages—rather, while computing PCIs, same parameters are estimated again based on one or more newly drawn samples. This sampling strategy makes the estimation procedure uneconomical and the situation worsens for processes requiring destructive tests or where, the cost of sampling is exorbitant.

To address these problems, Spiring (1995) used a fusion of the control chart techniques and the process capability analysis. The concept of process capability control chart was originally proposed by Boyles (1991) who designed a control chart for C_{pm} to observe variation in the observed C_{pm} values over the samples due to sampling fluctuation as well as due to various assignable causes. Later, Spiring (1995) modified this C_{pm} control chart by using the estimators C_{pm} based on the information already gathered by the control charts while checking stability of the process. This procedure is suitable for continuous monitoring of the performance of a process as it instantly reflects any change in the process prompting quick action from the concerned authority and eventually increases the quality level of the process and this makes it more economical. Moreover, process capability control charts require lesser sample size also (see Spiring, 1995) as compared to the so called "one shot" estimation of the process capability.

Despite providing such an useful methodology for continuous assessment of a process, Spiring's (1995) approach suffers from a basic problem which needs to be addressed. Based on our earlier discussion, the plug-in estimators of C_{pm} can be defined in the following two ways:

1. $\widehat{C}^*_{pm} = \frac{d}{3\sqrt{(S^2+(\overline{X}-T)^2)^2}}$, which is the usual plug-in estimator of C_{pm}; and
2. $\widehat{C}^{(S)}_{pm} = \frac{d}{3\sqrt{(\frac{\overline{S}}{c_4})^2+(\overline{\overline{X}}-T)^2}}$ or $\widehat{C}^{(R)}_{pm} = \frac{d}{3\sqrt{(\frac{\overline{R}}{d_2})^2+(\overline{\overline{X}}-T)^2}}$, depending on whether the parameters are estimated using information $\overline{X} - S$ chart or $\overline{X} - R$ chart, respectively.

Spiring (1995), while designing the control chart of C_{pm}, took the distribution of $\widehat{C}^*_{pm}$ into account, while in the final expressions of the control limits, he recommended to replace C_{pm} by $\widehat{C}^{(S)}_{pm}$ or $\widehat{C}^{(R)}_{pm}$ depending on the situation. This generates a conflict as the distribution is for single sample information while, both of $\widehat{C}^{(S)}_{pm}$ and $\widehat{C}^{(R)}_{pm}$ are based on multiple samples. As a result, the contribution of the number of subgroups (m) were never reflected in the corresponding statistical distribution. Therefore, Spiring's (1995) approach of constructing the estimated control limits of the process capability control chart needs some modification.

Again, Chen et al. (2007) designed process capability control charts for C_{PU} and C_{PL} as follows:

$$\left.\begin{aligned} \text{UCL} &= \frac{b_{n-1}}{3\sqrt{n}} \times t_{n-1,1-\alpha/2}(n-1, \delta = 3\sqrt{n}\,C_I) \\ \text{CL} &= C_I \\ \text{LCL} &= \frac{b_{n-1}}{3\sqrt{n}} \times t_{n-1,\alpha/2}(n-1, \delta = 3\sqrt{n}\,C_I) \end{aligned}\right\}, \tag{2}$$

where C_I stands for C_{PU} or C_{PL}, whichever is applicable, depending on the situation and $t_{n-1}(\delta)$ denotes the noncentral t distribution with $(n-1)$ degrees of freedom and non-centrality parameter δ. These control charts also suffer from similar problem as those of Spiring (1995).

In this article, we redefined the estimators of C_I based on information from both the $\overline{X} - S$ and $\overline{X} - R$ control charts. We also showed that the changes in the formulations of the estimators of C_I, due to the use of information gathered from the respective $\overline{X} - S$ or $\overline{X} - R$ control charts, indeed change the underlying statistical distribution (through the corresponding degrees of freedom) of these newly developed estimators and consequently, the control limits in Eq. (2) get changed as well.

In the next section, a brief review is done on the literature of control charts and various types of estimators of C_{PU} and C_{PL} along with their distributional aspects. A list of notations used in the succeeding chapters are enlisted in Sec. 3. The statistical distributions, unbiased estimators and the UMVUEs (wherever available) of the estimators of C_{PU} and C_{PL}, based on the information from the corresponding $\overline{X} - R$ and $\overline{X} - S$ charts, are formulated in Sec. 4. In Sec. 5, the corresponding process capability control charts are designed along with a brief discussion on the expressions for their operating characteristic (OC) curves, α and β risks and average run lengths (ARL). Two numerical examples are discussed in Sec. 6 to make a comparative study of the performances of the newly designed control charts with respect to Chen et al.'s (2007) control chart given in Eq. (2). Finally, the article concludes in Sec. 7 with a general note on the topics discussed in this article.

2. A Brief Literature Review on Control Charts and the Process Capability Indices C_{PU} and C_{PL}

Control chart is an online process monitoring technique used with a primary objective of quick detection of the occurrences of assignable causes of variation and to restrict the production of non-conforming items with respect to the preassigned specification limits. The original idea of control chart technique belongs to Shewhart (1931). The basic concept behind Shewhart's control chart thechnique is to identify two distinct types of variations that a process inherits viz., (i) chance causes of variation and (ii) assignable causes of variation. While the first type of variation is due to random fluctuation in the sample observations and hence is beyond control; the assignable causes of variation can, very well, be controlled. To monitor process centering over various rational subgroups, Shewhart (1931) proposed the so called $\overline{X}$ chart. Also, process variability can be monitored using R-chart or S-chart, based, respectively, on range or standard deviation. The mathematical expressions for these charts are available in any standard book on statistical quality control (for example, refer to Montgomery, 2000). All of these three control charts are based on the common assumption that the underlying distribution of the concerned quality characteristic is normal.

Due to their immense potential of practical applications, properties of these control charts have been extensively studied by a number of eminent researchers. Chen (1997) studied the distribution of the run length of $\overline{X}$ chart when the parameters (viz., μ and σ) of the quality characteristic and consequently control limits are estimated. Tsai et al. (2005) proposed control limits of $\overline{X}$ chart when the number of subgroups is small. This can be used to monitor the process mean at an earlier stage of production as compared to other existing approaches. Costa (1999) studied the performance of $\overline{X} - R$ chart to detect small to moderate shifts in mean and variability of the quality characteristic when the sample size varies over the subgroups and the values of the parameters are known. Castagliola et al. (2012) studied the performance (in terms of the run length) of $\overline{X}$ chart with variable sample size when the parameters are unknown and hence are estimated. Schilling and Nelson (1976) explored the effect of non-normality on the $\overline{X}$ chart. The authors showed the "manner of approach to normality" for various possible distributions of the concerned quality characteristics along with the respective sample sizes.

Process capability analysis is another important quality control tool used for assessing the capability of a process. Particularly, for processes with unilateral specification limits, most of the research articles available in literature are devoted to the indices C_{PU} and C_{PL}, defined in Eq. (1). Kane (1986) proposed the plug-in estimators of C_{PU} and C_{PL} as $\widehat{C}_{PU} = \frac{USL-\overline{X}}{3s}$ and $\widehat{C}_{PL} = \frac{\overline{X}-LSL}{3s}$, respectively, where, $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $s^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$.

Chou and Owen (1989) showed that under normality assumption, $\widehat{C}_{PU} \sim C_n t_{n-1}(\delta_U)$ and $\widehat{C}_{PL} \sim C_n t_{n-1}(\delta_L)$ where, $C_n = (3\sqrt{n})^{-1}$, $\delta_U = (3\sqrt{n})^{-1} C_{PU}$, $\delta_L = (3\sqrt{n})^{-1} C_{PL}$ and $t_{n-1}(\delta)$ is a non-central 't' distribution with (n−1) degrees of freedom and non-centrality parameter δ. Since both of these estimators are biased, Pearn and Chen (2002) derived the UMVUE's of C_{PU} and C_{PL} as $\widetilde{C_{PU}} = b_{n-1}\widehat{C_{PU}}$ and $\widetilde{C}_{PL} = b_{n-1}\widehat{C}_{PL}$ respectively, where, $b_{n-1} = \sqrt{\frac{2}{n-1}} \times \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})}$. They also suggested a step-by-step testing procedure for testing whether a process meets a pre-assigned minimum quality level, say, "C" as

$$H_0 : C_I \leq C$$

$$\text{against } H_1 : C_I > C$$

and derived expressions for the corresponding p-value, critical value and the power of the test.

Chou and Owen (1989) established lower confidence bounds (LCB) on C_{PU} and C_{PL} based on the natural estimators $\widehat{C}_{PU}$ and $\widehat{C}_{PL}$. However, since $\widehat{C}_{PU}$ and $\widehat{C}_{PL}$ estimators are biased, Pearn and Shu (2003) derived a $100\gamma\%$ LCB say, "C_U" and "C_L" of C_{PU} and C_{PL}, respectively, such that for quality characteristics of smaller the better type, $\gamma = P[t_{n-1}(\delta_1) \geq t_1]$, where $t_1 = -k_1\sqrt{n}, k_1 = 3\widehat{C}_{PU}, \delta_1 = -3\sqrt{n}C_U$ while for quality characteristics of larger the better type, $\gamma = P[t_{n-1}(\delta_2) \geq t_2]$, where $t_2 = k_2\sqrt{n}, k_2 = 3\widehat{C}_{PL}, \delta_2 = 3\sqrt{n}C_L$. Pearn and Liao (2006) studied the properties of the plug-in estimators of C_{PU} and C_{PL}, when the samples are contaminated by measurement error. Shu et al. (2006) defined plug-in estimators of C_{PU} and C_{PL} based on multiple samples and have explored the statistical properties of those estimators. Chatterjee and Chakraborty (2012) made an extensive review of the research articles dealing with PCIs for unilateral specification limits, in general.

3. List of Notations

Following is a list of notations and terminologies and their respective meanings or formulations (as the case may be), which will be used in the succeeding sections.

1. m = Number of rational sub-groups;
2. n = Sample size;
3. $N = m \times n$ is the total number of sample observations over all the subgroups;
4. c_4, d_2, and d_3 are usual constants of control charts (see Montgomery, 2000) and are functions of "n" only.
5. $d_2^* = (d_2^2 + \frac{d_3^2}{m})^{\frac{1}{2}}$;

6. $v = \frac{1}{-2+2\sqrt{1+2\times\frac{d_3^2}{m\times d_2^2}}}$;
7. $c = d_2 \times \sqrt{\frac{v_1}{2}} \times \frac{\Gamma\left(\frac{v_1}{2}\right)}{\Gamma\left(\frac{v_1+1}{2}\right)}$, where, v_1 is obtained by substituting $m = 1$ in the expression for 'v';
8. $b_n = \frac{\sqrt{\frac{2}{n}}\,\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$;
9. $\delta_U^{(N)} = 3\sqrt{N}C_{PU}$;
10. $\delta_L^{(N)} = 3\sqrt{N}C_{PL}$; and
11. $\delta_I^{(N)}$ is $\delta_U^{(N)}$ or $\delta_L^{(N)}$ depending upon the availability of USL or LSL, respectively;

4. Statistical Properties of the Estimators of C_{PU} and C_{PL} based on Information from Control Charts

Suppose the quality characteristic under consideration is normally distributed and also the concerned process is under statistical control. Also, suppose, while checking the stability of a process, we have "m" rational subgroups (see Montgomery, 2000) and from each rational subgroup, a sample of size "n" is drawn. Here we assume constant sample size for all the rational subgroups. So the total number of observations is $N = mn$. Also, let X_{ij} is the measured value of the quality characteristic for the j^{th} sample from the i^{th} rational subgroup. Then:

1. $\overline{\overline{X}} = \frac{1}{N}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij} = \frac{1}{m}\sum_{i=1}^{m}\overline{X}_i$ is the average of subgroup averages;
2. $\overline{S} = \frac{1}{m}\sum_{i=1}^{i=m} S_i$ is the average of the standard deviations over the subgroups with $S_i = \sqrt{\frac{1}{n-1}\sum_{j=1}^{n}(X_{ij} - \overline{X}_i)^2}$ being the standard deviation corresponding to the i^{th} subgroup for $i = 1(1)m$; and
3. $\overline{R} = \frac{1}{m}\sum_{i=1}^{i=m} R_i$ is the average of the ranges over the subgroups with R_i being the range corresponding to the i^{th} subgroup for $i = 1(1)m$.

Spiring (1995) argued that if the small sample properties of a PCI are known, then the control chart estimates of μ and σ, viz. $\overline{\overline{X}}$ and $\overline{S}/c_4$, respectively (when $\overline{X}$ - S chart is used) or $\overline{\overline{X}}$ and $\overline{R}/d_2$, respectively (when $\overline{X}$ - R chart is used) should be used to study the distributional properties of the estimated PCI's.

4.1. Distributions and Unbiased Estimators of C_{PU} and C_{PL} When Information is Gathered from the Corresponding $\overline{X} - R$ Control Charts

In the context of statistical quality control (SQC), especially for control charting, generally, the sample size is considerably small, say, 4–6 (see Montgomery, 2000). Hence, often range is used as a measure of dispersion and, consequently, $\overline{X} - R$ charts are the most commonly used control charts for checking the stability of a process. Woodall and Montgoemry (2000) have discussed two different estimators of σ, based on the average sample range $[\overline{R} = \frac{1}{m}\sum_{i=1}^{m} R_i]$, given by:

1. $\widehat{\sigma}_1 = \frac{\overline{R}}{d_2}$, which is generally used to design control chart and to analyze capability of a process; and
2. $\widehat{\sigma}_2 = \frac{\overline{R}}{d_2^*}$, which is generally used in gauge repeatability and reproducibility study.

Hence, $\frac{\overline{R}}{d_2}$ will be more appropriate estimator of σ in the present context. Woodall and Montgoemry (2000) made a thorough study of these two estimators of σ.

Following Spiring's (1995) approach, from Eq. (1), the estimator of C_{PU} can be redefined, by replacing μ and σ in Eq. (1) by their estimators $\overline{\overline{X}}$ and $\widehat{\sigma}_1 = \overline{R}/d_2$, respectively, as

$$\widehat{C}_{PU}^{(R)} = \frac{d_2}{3} \times \frac{U - \overline{\overline{X}}}{\overline{R}}. \tag{3}$$

The statistical distribution of $\widehat{C}_{PU}^{(R)}$ and the corresponding unbiased estimator are derived in the following theorem.

Theorem 4.1. *$\widehat{C}_{PU}^{(R)} \sim \frac{d_2}{3d_2^*\sqrt{N}} t_v(\delta_U^{(N)})$, with, $\delta_U^{(N)} = 3\sqrt{N}C_{PU}$, where, $t_v(\delta_U^{(N)})$ denotes the non central t-distribution with "v" degrees of freedom and the corresponding non centrality parameter is $\delta_U^{(N)}$.*

Also, $\widetilde{C}_{PU}^{(R)} = \frac{d_2^*}{d_2} \times b_v \times \widehat{C}_{PU}^{(R)}$ is an unbiased estimator of C_{PU} corresponding to the plug-in estimator $\widehat{C}_{PU}^{(R)}$, where, $b_v = \frac{\sqrt{\frac{2}{v}}\,\Gamma(\frac{v}{2})}{\Gamma(\frac{v-1}{2})}$.

Proof. Suppose, $X_{ij} \overset{iid}{\sim} N(\mu, \sigma^2)$. Then, $\overline{\overline{X}} \sim N(\mu, \sigma^2/N)$ and hence,

$$Z^* = \frac{\sqrt{N}(U - \overline{\overline{X}})}{\sigma} \sim N(\delta_U^{(N)}, 1). \tag{4}$$

Again, $\frac{\overline{R}}{\sigma} \sim d_2^* \frac{\chi_v}{\sqrt{v}}$ approximately (Woodall and Montgoemry, 2000), where "v" represents the fractional degrees of freedom for the χ−distribution. Kuo (2010) formulated the mathematical expression for "v".

Therefore, from Eq. (3):

$$\widehat{C}_{PU}^{(R)} = \frac{d_2}{3\sqrt{N}} \times \frac{\sigma}{\overline{R}} \times Z^*$$

$$\text{and hence, } \widehat{C}_{PU}^{(R)} \sim \frac{d_2}{3d_2^*\sqrt{N}} t_v(\delta_U^{(N)}) \tag{5}$$

$$\text{which implies, } E[\widehat{C}_{PU}^{(R)}] = \frac{d_2}{d_2^* \times b_v} \times C_{PU}$$

Thus,

$$\widetilde{C}_{PU}^{(R)} = \frac{d_2^* b_v}{d_2} \times \widehat{C}_{PU}^{(R)} \tag{6}$$

is an unbiased estimator of C_{PU}.

Hence, the proof.

Similarly, it is easy to show that

$$\widehat{C}_{PL}^{(R)} \sim \frac{d_2}{3d_2^*\sqrt{N}} t_v(\delta_L^{(N)})$$

Table 1
Relative Efficiency of Range Compared to Standard Deviation for Different Sample Sizes

Sample Size (n)	Relative Efficiency
2	1.000
3	0.992
4	0.975
5	0.955
6	0.930
10	0.850

and $\widetilde{C}^{(R)}_{PL} = \frac{d_2^* b_v}{d_2} \times \widehat{C}^{(R)}_{PL}$ is an unbiased estimator of C_{PL}.

Thus, in general,

$$\left.\begin{aligned} &\widehat{C}^{(R)}_{I} \sim \frac{d_2}{3d_2^*\sqrt{N}}\, t_v(\delta^{(N)}_{I}),\ \text{where, } \delta^{(N)}_{I} = 3\sqrt{N}C_I \\ &\text{and, } \widetilde{C}^{(R)}_{I} = \frac{d_2^* b_v}{d_2} \times \widehat{C}^{(R)}_{I} \ \text{ is an unbiased estimator of } C_I. \end{aligned}\right\} \tag{7}$$

□

4.2. *Distributions and UMVUEs of the Estimators of C_{PU} and C_{PL} when Information is Gathered from the Corresponding $\overline{X} - S$ Control Charts*

Due to the simplicity of computation as well as ease of interpretation, range is the most popular measure of dispersion among the practitioners. However, sometimes, despite these advantages, standard deviation outperforms range in measuring dispersion and consequently, $\overline{X} - S$ chart is used instead of $\overline{X} - R$ chart to check and establish stability of a process. Montgomery (2000) enlisted the following situations where standard deviation is preferred over range as a measure of dispersion:

(i) when the sample size is moderately large, say, $n > 10$; and
(ii) when the sample size is not constant.

Table 1 shows the tabulated values of the relative efficiency of range in comparison with standard deviation for various sample sizes (see Montgomery, 2000).

From Table 1 it is evident that for $n > 10$, range does not perform satisfactorily. Hence, similar to the formulation of the estimator of C_I based on information from $\overline{X} - R$ chart in Sec. 4.1, it is required to formulate the same based on information gathered from $\overline{X} - S$ chart as well. Such estimators are specially useful for processes with at least moderately large sample size.

The plug-in estimator of C_{PU} is obtained by replacing μ and σ in Eq. (1) by their estimators $\overline{\overline{X}}$ and $\overline{S}/c_4$, respectively, as obtained from the corresponding $\overline{\overline{X}} - S$ chart. Thus, the plug-in estimator of C_{PU}, based on the estimators of μ and σ as obtained from $\overline{X}$− S control chart, is

$$\widehat{C}^{(s)}_{PU} = \frac{c_4}{3} \times \frac{U - \overline{\overline{X}}}{\overline{S}}. \tag{8}$$

Under normality assumption, $\overline{S}$ is approximately normally distributed (Kirmani et al., 1991; Pearn and Kotz 2007). However, very little study has been done to construct the exact distribution of $\overline{S}$, although due to various economical and technical constraints, often, sample sizes observed in literature are not considerably large and hence inadequate for large sample approximation. Lemma 4.1 gives the exact distribution of $\overline{S}$ which will be particularly useful in the subsequent section on process capability control charts of C_I as there we need to know the exact distribution of the estimators of C_I, when the sample size is not considerably large.

Lemma 4.1. $\dfrac{\overline{S}^2}{\sigma^2} \sim \dfrac{1}{m(N-m)} \times \chi^2_{m(N-m)}$

Proof.

$$\frac{\overline{S}^2}{\sigma^2} = \frac{1}{m^2} \times \left(\frac{S_1 + S_2 + \cdots + S_m}{\sigma}\right)^2$$
$$= \frac{1}{m^2(n-1)} \times \left(\frac{a_1 + a_2 + \cdots + a_m}{\sigma}\right)^2, \text{ say}$$

where, $a_i = \sqrt{\sum_{j=1}^{n} (X_{ij} - \overline{X}_i)^2}$.

Now, $\left(\frac{a_i}{\sigma}\right)^2 \sim \chi^2_{n-1}$, $i = 1(1)m$ and since a_i's are independent and identically distributed (iid), $\left(\frac{a_i}{\sigma}\right) \times \left(\frac{a_i'}{\sigma}\right) \sim \chi^2_{n-1}$, for $i \neq i' = 1(1)m$.

Hence, due to the additive property of chi-square distribution, $\left(\frac{a_1+a_2+\cdots+a_m}{\sigma}\right)^2$ follows chi-square distribution with degrees of freedom $m(n-1) + m(m-1)(n-1) = m(N-m)$ i.e. $\frac{\overline{S}^2}{\sigma^2} \sim \frac{1}{m(N-m)} \times \chi^2_{m(N-m)}$ and hence the proof.

Now, using Lemma 4.1, the underlying statistical distribution of $\widehat{C}^{(s)}_{PU}$ and the corresponding UMVUE are derived in the following results. □

Theorem 4.2. *$\widehat{C}^{(S)}_{PU} \sim \frac{c_4}{3\sqrt{N}} t_{m(N-m)}(\delta^{(N)}_U)$, with $\delta^{(N)}_U = 3\sqrt{N} C_{PU}$, where, $t_{m(N-m)}(\delta^{(N)}_U)$ denotes the non-central t-distribution with $m(N-m)$ degrees of freedom and the corresponding non centrality parameter is $\delta^{(N)}_U$.*

Also, $\widetilde{C}^{(s)}_{PU} = \frac{b_{m(N-m)}}{c_4} \times \widehat{C}^{(s)}_{PU}$ is an unbiased estimator of C_{PU} corresponding to the plug-in estimator $\widehat{C}^{(S)}_{PU}$.

Proof. From Eqs. (8) and (4),

$$\widehat{C}^{(S)}_{PU} = \frac{c_4}{3\sqrt{N}} \times \frac{Z^*}{\sqrt{\frac{\overline{S}^2}{\sigma^2}}}$$

$$\text{and hence using Lemma 4.1, } \widehat{C}^{(S)}_{PU} \sim \frac{c_4}{3\sqrt{N}} \times t_{m(N-m)}(\delta^{(N)}_U). \tag{9}$$

$$\text{Thus, } E[\widehat{C}^{(S)}_{PU}] = \frac{c_4}{b_{m(N-m)}} \times C_{PU}$$

and $\widetilde{C}_{PU}^{(S)} = \frac{b_{m(N-m)}}{c_4} \times \widehat{C}_{PU}^{(S)}$ is an unbiased estimator of C_{PU}.

Hence, the proof.

Similarly,

$$\widehat{C}_{PL}^{(S)} \sim \frac{c_4}{3\sqrt{N}} \times t_{m(N-m)}(\delta_L^{(N)}).$$

Also, $\widetilde{C}_{PL}^{(S)} = \frac{b_{m(N-m)}}{c_4} \times \widehat{C}_{PL}^{(S)}$ is the unbiased estimator of C_{PL}.

Thus, in general,

$$\left.\begin{array}{l} \widehat{C}_I^{(S)} \sim \frac{c_4}{3\sqrt{N}} \times t_{m(N-m)}(\delta_I^{(N)}) \\ \text{and, } \widetilde{C}_I^{(S)} = \frac{d_2^* b_v}{d_2} \times \widehat{C}_I^{(S)} \text{ is an unbiased estimator of } C_I. \end{array}\right\} \quad (10)$$

Our next objective is to check whether these unbiased estimators can be considered as the minimum variance unbiased estimators (UMVUE) as well. Since we have already assumed that $X_{ij} \sim N(\mu, \sigma^2)$, for $i = 1(1)m$, $j = 1(1)n$ and normal distribution belongs to the exponential family of distributions; $(\overline{\overline{X}}, \overline{S}^2)$ are jointly complete sufficient for (μ, σ^2) and hence, following Rao - Blackwell theorem (see Casella and Berger, 2007), $\widetilde{C}_{PU}^S$ and $\widetilde{C}_{PL}^S$ are UMVUEs of C_{PU} and C_{PL}.

Since $\overline{R}$ is not a function of $\sum_{i=1}^m \sum_{j=1}^n X_{ij}^2$, $(\overline{\overline{X}}, \frac{\overline{R}}{d_2})$ are not jointly complete sufficient statistics for (μ, σ^2) and consequently, neither of $\widetilde{C}_{PU}^{(R)}$ or $\widetilde{C}_{PL}^{(R)}$ are UMVUEs of C_{PU} or C_{PL}, respectively. □

5. Process Capability Control Charts of C_{PU} and C_{PL}

It has been already observed in Sec. 4 that the underlying statistical distribution of both $\widetilde{C}_I^{(R)}$ and $\widetilde{C}_I^{(S)}$ is non central t. Hence, to construct their process capability control charts, we can proceed as follows:

1. using probabilistic approach of control chart (similar to Chen et al., 2007; Spiring, 1995; and so on); and
2. constructing conventional control chart for non-normal (in particular t-distributed) variables (see Montgomery, 2000).

Since the process capability control charts designed so far are based on the probabilistic approach we also follow the same procedure to make our formulation comparable to the existing process capability control charts.

5.1. Process Capability Control Chart of C_{PU} and C_{PL} based on Information from the Corresponding $\overline{X} - R$ Control Chart

From Theorem 4.1, $\widehat{C}_{PU}^{(R)} \sim \frac{d_2}{3d_2^*\sqrt{N}} t_v(\delta_U^{(N)})$. Also, $\widetilde{C}_{PU}^{(R)} = \frac{d_2^* b_v}{d_2} \times \widehat{C}_{PU}^{(R)}$ is an unbiased estimator of C_{PU} (see Eq. (6)). Thus,

$$P\left[\frac{d_2}{3d_2^*\sqrt{N}} t_{v,\frac{\alpha}{2}}(\delta_U^{(N)}) \le \widehat{C}_{PU}^{(R)} \le \frac{d_2}{3d_2^*\sqrt{N}} \times t_{v,(1-\frac{\alpha}{2})}(\delta_U^{(N)})\right] = 1 - \alpha$$

$$\text{i.e. } P\left[\frac{b_v}{3\sqrt{N}} t_{v,\frac{\alpha}{2}}(\delta_U^{(N)}) \le \widetilde{C}_{PU}^{(R)} \le \frac{b_v}{3\sqrt{N}} t_{v,(1-\frac{\alpha}{2})}(\delta_U^{(N)})\right] = 1 - \alpha$$

and hence the corresponding control limits for the process capability control chart of C_{PU} will be

$$\left.\begin{aligned} \mathrm{UCL}_{C_{PU}^{(R)}} &= \frac{b_v}{3\sqrt{N}}\, t_{v,(1-\frac{\alpha}{2})}(\delta_U^{(N)}) \\ \mathrm{CL}_{C_{PU}^{(R)}} &= C_{PU} \\ \mathrm{LCL}_{C_{PU}^{(R)}} &= \frac{b_v}{3\sqrt{N}}\, t_{v,\frac{\alpha}{2}}(\delta_U^{(N)}) \end{aligned}\right\} \tag{11}$$

However, control limits in Eq. (11) are functions of the parameters μ and σ of the quality characteristics and in practice, their values are often unknown. Following Spiring (1995), we use the information gathered from the corresponding $\overline{X} - R$ chart and estimate μ and σ by $\overline{\overline{X}}$ and $\frac{\overline{R}}{d_2}$, respectively. Consequently, C_{PU} is replaced by $\overline{\widetilde{C}}_{PU}^{(*R)}$ and $\delta_U^{(N)}$ by $\overline{\widetilde{\delta}}_U^{(N,R)}$ in Eq. (11), where $\overline{\widetilde{C}}_{PU}^{(*R)} = \frac{1}{m}\sum_{i=1}^{m} \widetilde{C}_{PU_i}^{(*R)}$ with $\widetilde{C}_{PU_i}^{(*R)}$ being the value of the unbiased estimator of C_{PU} for the i^{th} rational subgroup with i = 1(1)m, (the expression of which is developed in Theorem 5.1 below) and the observed non-centrality parameter of the corresponding t- distribution is $\overline{\widetilde{\delta}}_U^{(N,R)} = 3\sqrt{N}\overline{\widetilde{C}}_{PU}^{(*R)}$. Thus, the modified control limits of the process capability control chart of C_{PU}, based on the information from $\overline{X} - R$ control chart, are obtained as

$$\left.\begin{aligned} \mathrm{UCL}_{\widetilde{C}_{PU}^{(R)}} &= \tfrac{b_v}{3\sqrt{N}}\, t_{v,(1-\frac{\alpha}{2})}(\overline{\widetilde{\delta}}_U^{(N,R)}) \\ \mathrm{CL}_{\widetilde{C}_{PU}^{(R)}} &= \overline{\widetilde{C}}_{PU}^{(*R)} \\ \mathrm{LCL}_{\widetilde{C}_{PU}^{(R)}} &= \tfrac{b_v}{3\sqrt{N}}\, t_{v,\frac{\alpha}{2}}(\overline{\widetilde{\delta}}_U^{(N,R)}) \end{aligned}\right\}. \tag{12}$$

Similarly, the process capability control chart for C_{PL} will be

$$\left.\begin{aligned} \mathrm{UCL}_{\widetilde{C}_{PL}^{(R)}} &= \tfrac{b_v}{3\sqrt{N}}\, t_{v,(1-\frac{\alpha}{2})}(\overline{\widetilde{\delta}}_L^{(N,R)}) \\ \mathrm{CL}_{\widetilde{C}_{PL}^{(R)}} &= \overline{\widetilde{C}}_{PL}^{(*R)} \\ \mathrm{LCL}_{\widetilde{C}_{PL}^{(R)}} &= \tfrac{b_v}{3\sqrt{N}}\, t_{v,\frac{\alpha}{2}}(\overline{\widetilde{\delta}}_L^{(N,R)}) \end{aligned}\right\}, \tag{13}$$

where, $\overline{\widetilde{C}}_{PL}^{(*R)} = \frac{1}{m}\sum_{i=1}^{m} \widetilde{C}_{PL_i}^{(*R)}$ with $\widetilde{C}_{PL_i}^{(R)}$ being the value of the unbiased estimator of C_{PL} for the i^{th} rational subgroup and $\overline{\widetilde{\delta}}_L^{(N,R)} = 3\sqrt{N}\overline{\widetilde{C}}_{PL}^{(*R)}$.

Hence, in general, the process capability control chart for C_I can be designed as

$$\left.\begin{aligned} \mathrm{UCL}_{\widetilde{C}_{I}^{(R)}} &= \tfrac{b_v}{3\sqrt{N}}\, t_{v,(1-\frac{\alpha}{2})}(\overline{\widetilde{\delta}}_I^{(N,R)}) \\ \mathrm{CL}_{\widetilde{C}_{I}^{(R)}} &= \overline{\widetilde{C}}_{I}^{(*R)} \\ \mathrm{LCL}_{\widetilde{C}_{I}^{(R)}} &= \tfrac{b_v}{3\sqrt{N}}\, t_{v,\frac{\alpha}{2}}(\overline{\widetilde{\delta}}_I^{(N,R)}) \end{aligned}\right\}. \tag{14}$$

Now, for any control chart, say $\overline{X}-$ chart, the usual convention is to construct the control limits in terms of the overall average $(\overline{\overline{X}})$ while the values, which are plotted in that control chart, are the averages $(\overline{X}_i)$ corresponding to individual subgroups, for $i = 1(1)m$. Therefore, it is required to develop the single sample unbiased estimator of C_{PU_i} when the corresponding plug-in estimator is given by $\widehat{C}_{PU_i}^{(*R)} = \frac{d_2(U-\overline{X}_i)}{3R_i}$ as this will be required while plotting the unbiased estimators of C_{PU} for individual subgroups in the process capability control chart. Here, $\overline{X}_i$ and R_i denote the average and range corresponding to the i^{th} subgroup, for $i = 1(1)m$. The expression for this unbiased estimator is developed in Theorem 5.1. Here, for notational simplicity, $\widehat{C}^{(*R)}_{PU_i}$, $\overline{X}_i$ and R_i are replaced by $\widehat{C}^{(*R)}_{PU}$, $\overline{X}$, and R, respectively.

Theorem 5.1. $\widehat{C}^{*(R)}_{PU} \sim \frac{d_2}{3c\sqrt{n}} t_{v_1}(\delta^{(n)}_U)$.

Also, the corresponding unbiased estimator of C_{PU} *is* $\widetilde{C}^{(*R)}_{PU} = \frac{c\, b_{v_1}}{d_2} \times \frac{U - \overline{X}}{3R}$.

Proof. $\frac{R}{\sigma} \sim \frac{c}{\sqrt{v_1}} \chi_{v_1}$ (see Pearson, 1952; Spiring, 1995). Also, if $X_i \sim N(\mu, \sigma^2)$ for i = 1(1)n, then, $\sqrt{n}(\frac{U-\overline{X}}{\sigma}) \sim N(\delta^{(n)}_U, 1)$. Thus,

$$\widehat{C}^{*R}_{PU} \sim \frac{d_2}{3c\sqrt{n}} \times t_{v_1}(\delta^{(n)}_U) \tag{15}$$

(16)

Thus,

$$\widetilde{C}^{(*R)}_{PU} = \frac{cb_{v_1}}{d_2} \times \widehat{C}^{(*R)}_{PU} \tag{17}$$

is an unbiased estimator of C_{PU}.

Hence, the proof.

Similarly,

$$\widehat{C}^{*(R)}_{PL} \sim \frac{d_2}{3c\sqrt{n}} t_{v_1}(\delta^{(n)}_L)$$

Also,

$$\widetilde{C}^{(*R)}_{PL} = \frac{cb_{v_1}}{d_2} \times \widehat{C}^{(*R)}_{PL} \tag{18}$$

is an unbiased estimator of C_{PL}.

Here also neither of $\widetilde{C}^{(*R)}_{PU}$ or $\widetilde{C}^{(*R)}_{PL}$ are UMVUEs of C_{PU} or C_{PL}, respectively. □

5.2. Process Capability Control Chart of C_{PU} and C_{PL} based on Information from the Corresponding $\overline{X} - S$ Control Chart

From Theorem 4.2, $\widehat{C}^{(S)}_{PU} \sim \frac{c_4}{3\sqrt{N}}\, t_{m(N-m)}(\delta^{(N)}_U)$. Moreover, in Sec. 4.2, we have seen that $\widetilde{C}^{(s)}_{PU} = \frac{b_{m(N-m)}}{c_4} \times \widehat{C}^{(s)}_{PU}$ is an UMVUE of C_{PU}. Thus,

$$P\left[\frac{c_4}{3\sqrt{N}} \times t_{m(N-m),\frac{\alpha}{2}}(\delta^{(N)}_U) \le \widehat{C}^{(S)}_{PU} \le \frac{c_4}{3\sqrt{N}} \times t_{m(N-m),(1-\frac{\alpha}{2})}(\delta^{(N)}_U)\right] = 1-\alpha,$$

$$\text{i.e., } P\left[\frac{b_{m(N-m)}}{3\sqrt{N}} \times t_{m(N-m),\frac{\alpha}{2}}(\delta^{(N)}_U) \le \widetilde{C}^{(S)}_{PU} \le \frac{b_{m(N-m)}}{3\sqrt{N}} \times t_{m(N-m),(1-\frac{\alpha}{2})}(\delta^{(N)}_U)\right] = 1-\alpha.$$

Hence, the control limits of the process capability control chart of C_{PU}, based on the information from $\overline{X} - S$ chart, are given by

$$\left.\begin{array}{l} \mathrm{UCL}_{C_{PU}} = \frac{b_{m(N-m)}}{3\sqrt{N}} \times t_{m(N-m),(1-\frac{\alpha}{2})}(\delta_U^{(N)}) \\ \mathrm{CL}_{C_{PU}} = C_{PU} \\ \mathrm{LCL}_{C_{PU}} = \frac{b_{m(N-m)}}{3\sqrt{N}} \times t_{m(N-m),\frac{\alpha}{2}}(\delta_U^{(N)}) \end{array}\right\}, \tag{19}$$

Following Spiring (1995), here we use the information gathered from the corresponding $\overline{X} - S$ chart and estimate μ and σ by $\overline{\overline{X}}$ and $\frac{\overline{S}}{c_4}$, respectively. Consequently, C_{PU} is replaced by $\overline{\widetilde{C}}_{PU}^{(*S)}$ and $\delta_U^{(N)}$ by $\overline{\widetilde{\delta}}_U^{(N,S)}$ in Eq. (19), where, $\overline{\widetilde{C}}_{PU}^{(*S)} = \frac{1}{m}\sum_{i=1}^{m} \widetilde{C}_{PU_i}^{(*S)}$ with $\widetilde{C}_{PU_i}^{(*S)}$ being the value of the UMVUE of C_{PU} for the i^{th} rational subgroup with i = 1(1)m and the observed non centrality parameter of the corresponding noncentral t- distribution is $\overline{\widetilde{\delta}}_U^{(N,S)} = 3\sqrt{N}\overline{\widetilde{C}}_{PU}^{(*S)}$.

In this context, Pearn and Chen (2002) formulated the UMVUE of C_{PU} for single sample information when σ^2 is replaced by S^2. Suppose the plug-in estimator of C_{PU}, based on single sample information, is defined as $\widehat{C}_{PU}^{(*S)} = \frac{U-\overline{X}}{3S}$. Then, the authors have shown that $\widetilde{C}_{PU}^{(*S)} = b_{n-1}\widehat{C}_{PU}^{(*S)}$ is the UMVUE of C_{PU}.

Thus, the modified control limits of the process capability control chart of C_{PU}, based on the information from $\overline{X} - S$ control chart, are obtained as

$$\left.\begin{array}{l} \mathrm{UCL}_{\widetilde{C}_{PU}^{(S)}} = \frac{b_{m(N-m)}}{3\sqrt{N}} \times t_{m(N-m),(1-\frac{\alpha}{2})}(\overline{\widetilde{\delta}}_U^{(N,S)}) \\ \mathrm{CL}_{\widetilde{C}_{PU}^{(S)}} = \overline{\widetilde{C}}_{PU}^{(*S)} \\ \mathrm{LCL}_{\widetilde{C}_{PU}^{(S)}} = \frac{b_{m(N-m)}}{3\sqrt{N}} \times t_{m(N-m),\frac{\alpha}{2}}(\overline{\widetilde{\delta}}_U^{(N,S)}) \end{array}\right\}. \tag{20}$$

Similarly, the process capability control chart for C_{PL}, based on information from the corresponding $\overline{X} - S$ control chart, will be

$$\left.\begin{array}{l} \mathrm{UCL}_{\widetilde{C}_{PL}^{(S)}} = \frac{b_{m(N-m)}}{3\sqrt{N}} \times t_{m(N-m),(1-\frac{\alpha}{2})}(\overline{\widetilde{\delta}}_L^{(N,S)}) \\ \mathrm{CL}_{\widetilde{C}_{PL}^{(S)}} = \overline{\widetilde{C}}_{PL}^{(*S)} \\ \mathrm{LCL}_{\widetilde{C}_{PL}^{(S)}} = \frac{b_{m(N-m)}}{3\sqrt{N}} \times t_{m(N-m),\frac{\alpha}{2}}(\overline{\widetilde{\delta}}_L^{(N,S)}) \end{array}\right\}, \tag{21}$$

where $\overline{\widetilde{C}}_{PL}^{(*S)} = \frac{1}{m}\sum_{i=1}^{m} \widetilde{C}_{PL_i}^{(*S)}$ with $\widetilde{C}_{PL_i}^{(*S)}$ being the value of the UMVUE of C_{PU} for the i^{th} rational subgroup with $i = 1(1)m$ and $\overline{\widetilde{\delta}}_L^{(N,S)} = 3\sqrt{N}\overline{\widetilde{C}}_{PL}^{(*S)}$.

Hence, in general, process capability control chart of C_I can be designed, based on information from the corresponding $\overline{X} - S$ chart, as

$$\left.\begin{array}{l} \mathrm{UCL}_{\widetilde{C}_{I}^{(S)}} = \frac{b_{m(N-m)}}{3\sqrt{N}} \times t_{m(N-m),(1-\frac{\alpha}{2})}(\overline{\widetilde{\delta}}_I^{(N,S)}) \\ \mathrm{CL}_{\widetilde{C}_{I}^{(S)}} = \overline{\widetilde{C}}_{I}^{(*S)} \\ \mathrm{LCL}_{\widetilde{C}_{I}^{(S)}} = \frac{b_{m(N-m)}}{3\sqrt{N}} \times t_{m(N-m),\frac{\alpha}{2}}(\overline{\widetilde{\delta}}_I^{(N,S)}) \end{array}\right\} \tag{22}$$

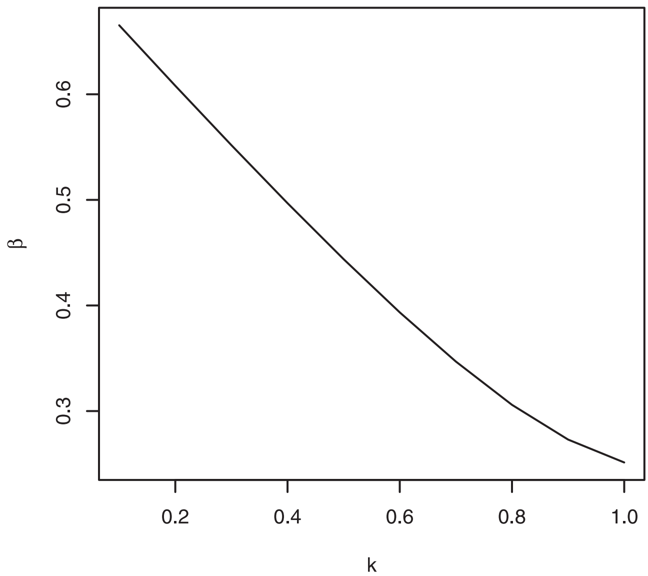

Figure 1. OC curve for CI control chart based on information from X-bar and R chart.

5.3. Operating Characteristic (OC) Curve and Average Run Length (ARL) of Process Capability Control Charts for C_{PU} and C_{PL}

From the point of view of interpretation as well as acceptability among the users, OC- curve and the average run length (ARL) are some of the very crucial performance yardsticks of a control chart. Montgomery (2000) defined OC as the ability of a control chart to detect a shift in the value of the corresponding quality characteristic and gives a graphical representation of the β risk against the magnitude of shift (k). The OC curve for the process capability control chart of C_I based on information from the corresponding $\overline{X} - R$ chart (see Eq. (14) with $C_{I_0} = 1.00$, $n = 5$, and $m = 6$ is given in Fig. 1.

Similarly, with information from $\overline{X} - S$ chart and $n = 11$, the corresponding OC curve for the process capability control chart of C_I (see Eq. (22)) is given in Fig. 2.

To study the impacts of k, m, n, and C_{I_0} on ARL, we have considered $k = 0.1(0.1)1$, $C_{I_0} = 1.0(0.2)2.0$ and $m = 6$ along with two different values of n, viz., $n = 5$, $n = 11$. Therefore, while constructing process capability control chart of C_I, information obtained from $\overline{X} - R$ control chart will be applicable for the first case; while for the second case, information from $\overline{X} - S$ control chart will be useful.

Thus, when $n = 11$, although the $\beta-$ values vary for various values of k and C_{I_0}, the value of ARL remains 2 irrespective of the values of k and C_{I_0}. However, when $n = 5$, ARL values differ to a large extent. This is also quite justified as higher sample size makes a control chart more robust towards sampling fluctuation and hence ARL should ideally be smaller. These values of ARL are given in Table 2.

The following can be observed from Table 2.

1. When the shift (k) is fixed, $\beta-$ risk and ARL increases with the increase in C_{I_0} value.
2. For fixed C_{I_0}, ARL decreases with the increase in k. This implies that the newly designed process capability control charts are more efficient to larger shifts than the smaller ones. They have inherited this characteristic from the corresponding $\overline{X}$ chart.

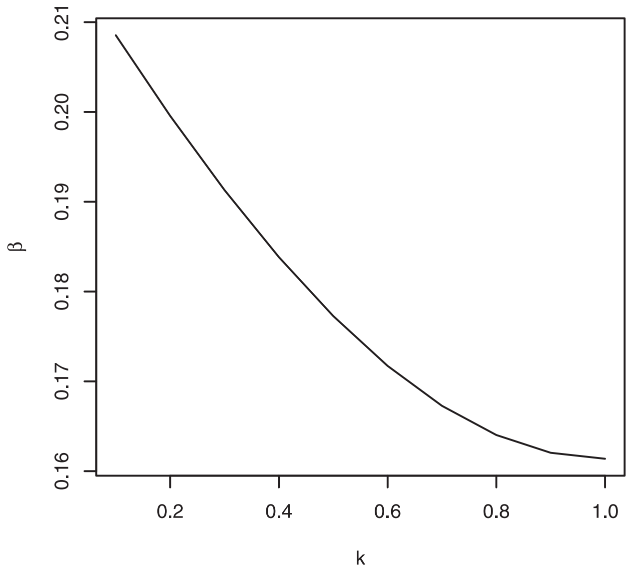

Figure 2. OC curve for CI control chart based on information from X-bar and S chart.

Although ARL remains constant for C_I control chart with information from $\overline{X} - S$ chart and various combinations of k and C_{I_0} values; similar observations, as listed above, also hold good for their β-risks.

In this context, the $\beta-$ risk and ARL of Chen et al.'s (2007) limits are higher than those of our proposed C_I control charts of both the types. This is due to the fact that, since the degrees of freedom (see, $m(N - m)$ or "v" as the case may be) corresponding to the noncentral t distribution attached to the control limits of the proposed C_I chart is greater than that corresponding to Chen et al.'s (2007) (see, $(n - 1)$), the $\beta-$ value and ARL for the proposed control chart will always be smaller than that of Chen et al.'s (2007) chart.

Table 2
ARL Values for $m = 6$, $n = 5$, $k = 0.1(0.1)1$ and $C_{I_0} = 1.0(0.1)2.0$

	C_{I_0}					
k	1.00	1.20	1.40	1.60	1.80	2.00
0.1	3	4	7	26	–	–
0.2	3	3	5	11	–	–
0.3	2	3	4	7	26	–
0.4	2	3	3	5	11	–
0.5	2	2	3	4	7	26
0.6	2	2	3	3	5	11
0.7	2	2	2	3	4	7
0.8	2	2	2	3	4	5
0.9	2	2	2	2	3	4
1.0	2	2	2	2	3	3

Table 3
Data and $\widehat{C}_{PU}^{*(R)}$ Values of 'X' for Various Subgroups

Sample No.	Sub-group 1	Sub-group 2	Sub-group 3	Sub-group 4	Sub-group 5	Sub-group 6
1	0.17	0.16	0.13	0.19	0.13	0.17
2	0.17	0.16	0.15	0.14	0.18	0.19
3	0.16	0.18	0.17	0.13	0.16	0.15
4	0.19	0.17	0.14	0.16	0.12	0.14
5	0.14	0.13	0.19	0.17	0.13	0.16
$\widetilde{C}_{PU}^{(*R)}$	**1.7638**	**1.8428**	**1.5794**	**1.5576**	**1.7112**	**1.8164**

For a control chart, ARL is a "smaller the better" type of characteristic in a sense that it will require less number of samples before detecting a shift in the process (Montgomery, 2000). Our proposed C_I control charts out-perform Chen et al.'s (2007) control charts with respect to ARL.

6. Numerical Examples

6.1. Example 1

The data for this example is from a chemical manufacturing industry in Mumbai, India. The quality characteristic under consideration is coded as "X" and it is of smaller the better type with corresponding USL given by 0.3 unit. In the present example, $n = 5, m = 6, N = 30$ and $USL = 0.3$. Also, for $n = 5$, $c_4 = 0.94, d_2 = 2.326, d_3 = 0.864$. Here, $v = 21.9899 \approx 22, v_1 = 3.8586 \approx 4, c = 2.4745$ and hence $b_v = 0.9654$ and $b_{v_1} = 0.7979$. Since $n < 10$, we shall design the C_{PU} control chart based on information from the corresponding $\overline{X} - R$ control chart. The data along with the observed values of $\widetilde{C}_{PU}^{(R)}$ for individual subgroups are given in Table 3 below.

From Table 3, $\overline{\overline{X}} = 0.1577, \overline{R} = 0.055, \overline{\widetilde{C}}_{PU}^{(*R)} = 1.7119$, and $\overline{\widetilde{\delta}}^{(N,R)} = 28.1294$. Hence, when $\alpha = 0.05$, Eq. (12) gives

$$\left.\begin{array}{l} \mathrm{UCL}_{\widetilde{C}_{PU}^{(R)}} = 2.3537 \\ \mathrm{CL}_{\widetilde{C}_{PU}^{(R)}} = 1.7119 \\ \mathrm{LCL}_{\widetilde{C}_{PU}^{(R)}} = 1.2655 \end{array}\right\} \tag{23}$$

The corresponding C_{PU} control chart together with the corresponding $\widetilde{C}_{PU}^{*(R)}$ values are given in Fig. 3.

From Fig. 3, it can be observed that all the $\widetilde{C}_{PU}^{*(R)}$ values lie within $UCL_{C_{PU}^{(R)}}$ and $LCL_{C_{PU}^{(R)}}$ and indicate consistently high capability of the process. This consistency in the capability level of the process allows us to summarize the overall capability of the process through $\widetilde{C}_{PU}^{(R)}$.

Here, $d_2^* = 2.3490$ and $\widehat{C}_{PU}^{(R)} = 2.006$. Hence, $\widetilde{C}_{PU}^{(R)} = \frac{d_2^* \times b_v}{d_2} \times \widehat{C}_{PU}^{(R)} = 1.9556$ which indicates that the overall capability of the process is really very high. Here, $\widetilde{C}_{PU}^{(R)}$ is not the mere average of individual UMVUEs as is evident from Table 3, since all the $\widetilde{C}_{PU}^{(*R)}$ values for individual subgroups are less than this value.

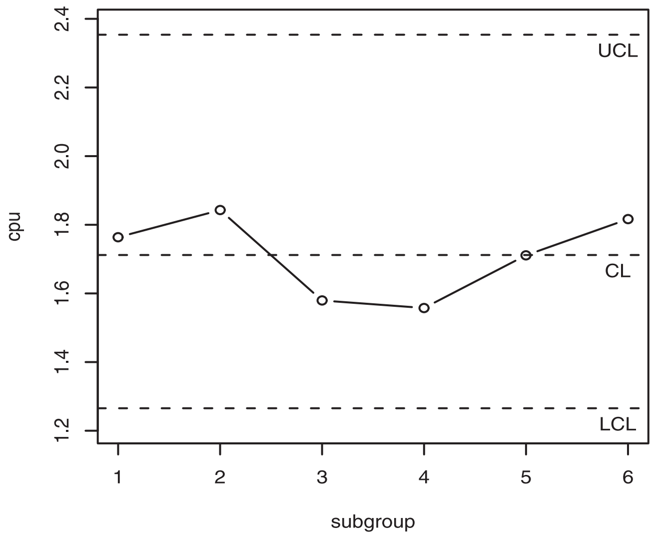

Figure 3. CPU control chart based on information from X-bar and R chart for data in Table 3.

In this context, for the present data set, the control limits, based on Chen et al.'s (2007) approach, are as follows:

$$\left.\begin{array}{l} \mathrm{UCL}_{C_{PU}}^{(Chen)} = 4.1188 \\ \mathrm{CL}_{C_{PU}}^{(Chen)} = 1.7809 \\ \mathrm{LCL}_{C_{PU}}^{(Chen)} = 0.8270 \end{array}\right\}. \tag{24}$$

Note that although for this particular example, the inference about the performance of the process, over various subgroups, remains the same for both the approaches, the control limits using Chen et al.'s (2007) approach are much more wider compared to our proposed limits. Hence use of their control limits increase the chance of overlooking some serious drawbacks in the process. In fact, the higher ARL values corresponding to Chen et al. (2007) limits, as compared to our newly proposed control limits, as discussed in Sec. 5, also justify this phenomenon. This can be visualized more clearly in the next example.

6.2. Example 6.2

Here we use a data collected from a integrated circuit (IC) manufacturing process. This data was originally used by Chen et al. (2007). In IC manufacturing and packaging process, one of the major quality characteristics is wire bonding of gold wire as it plays a critical role in the normal operation of IC. This is a quality characteristic of higher the better type. The data set consists of 1.2 mm diameter type gold wire with measurements of wire strength (in grams) in the pull test.

Here $m = 25, n = 11$, i.e., $N = 275$ and $LSL = 5$. Moreover, since $n > 10$, while constructing process capability control chart of C_{PL}, the relevant information is to be gathered from the corresponding $\overline{X} - S$ chart. Since here $m(N - m) = 6250$ is sufficiently large, $b_{m(N-m)} \approx 1$. Also, here $\overline{\overline{X}} = 8.0062$ and $\overline{S} = 0.8077$.

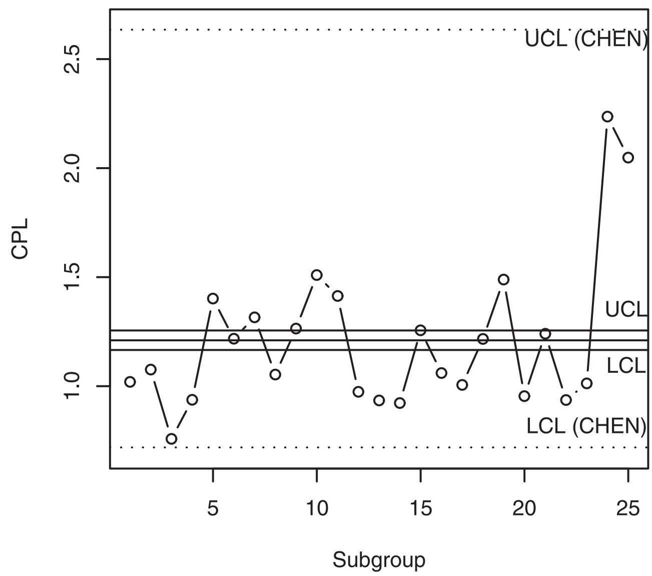

Figure 4. CPL control chart based on X-bar and S chart for Chen et al.'s (2007) data.

Now, for the present data, $\overline{\widetilde{C}}_{PL}^{(*S)} = 1.2102$ and $\overline{\widetilde{\delta}}^{(N,S)} = 760.2067$. Hence, when $\alpha = 0.05$, from Eq. (13), the control limits of the C_{PL} control chart will be:

$$\left.\begin{array}{l} \mathrm{UCL}_{\widetilde{C}_{PL}^{(S)}} = 1.2554 \\ \mathrm{CL}_{\widetilde{C}_{PL}^{(S)}} = 1.2102 \\ \mathrm{LCL}_{\widetilde{C}_{PL}^{(S)}} = 1.1659 \end{array}\right\} \tag{25}$$

The control limits of Eq. (25) for C_{PL} control chart are shown in Fig. 4 below along with the observed $\widetilde{C}_{PL}^{*(S)}$ values for various samples. The limits obtained by Chen et al. (2007), using Eq. (2), are also incorporated in this figure.

In Fig. 4, the control limits corresponding to Eq. (25) are drawn in continuous lines; while the dotted lines stand for Chen et al.'s (2007) limits. From this figure it can be seen that 13 out of the 25 points lie below $LCL_{\widetilde{C}_{PL}^{(S)}}$, where as 7 points lie above $UCL_{\widetilde{C}_{PL}^{(S)}}$. Although exceeding $UCL_{\widetilde{C}_{PL}^{(S)}}$ in general signifies producing items with better quality under the given set-up - in the present context, incidence of having 20 out of 25 observations beyond either of the two control limits strongly suggest that the process is not having consistent capability. Thus, despite being stable as measured by the corresponding $\overline{X} - S$ chart, the process fails to perform consistently. In fact, the process sufferers from considerably high fluctuation in variability as measured over the samples from various subgroups. Although the corresponding S-chart fails to raise alarm for such fluctuations, it becomes evident as soon as the C_{PL} control chart is drawn. Also, it is not advisable to assess the overall capability of the process by using a single PCI under such circumstances as that may not represent the actual capability scenario of the process.

In this context, Chen et al.'s (2007) limits, in Fig. 4, ignore this inconsistency in the C_{PL} values at subgroup levels. Even, for the last two subgroups, where, the $\widetilde{C}_{PL}^{(*S)}$ values are considerably high, Chen et al.'s (2007) control chart still consider these points to be below the corresponding upper control limit.

7. Concluding Remarks

In this article, we have studied the underlying statistical distributions of the plug-in (natural) estimators of C_{PU} and C_{PL}, when the parameters (see μ and σ) of the concerned quality characteristic are estimated based on information drawn from the corresponding $\overline{X} - S$ or $\overline{X} - R$ charts. We also defined the unbiased estimators and UMVUEs (wherever applicable) for these estimators. Then, these estimators have been used to design process capability control charts of C_{PU} and C_{PL}. As a result, the conclusion drawn about the process regarding its overall capability is based on samples collected over a considerable period of time and hence is more robust in some sense towards the degree of fluctuation in the performance of a process from subgroup to subgroup. We also observed that although the original underlying distribution (noncentral t- distribution) remains the same, the degrees of freedom and the noncentrality parameter get changed as soon as we incorporate subgroup information in the C_{PU} or C_{PL} control chart. We also noted that even statistically stable processes may have inconsistent capability (as in Example 2), i.e., statistical stability does not ensure consistency in the quality of the items produced. Absence of consistency in a process may lead to confusion if we try to evaluate the overall capability of the process through a single capability index. Thus, along with the $\overline{X} - S$ or $\overline{X} - R$ charts, one should also construct the corresponding process capability control chart before assessing the overall capability of the process.

We saw, through two numerical examples, that our newly designed process capability control charts for C_{PU} and C_{PL} perform better than the existing charts. However, as can be seen from Example 2, when at least one of m and n become considerably high (here $m = 25$), the degrees of freedom of the corresponding non-central t- distribution get highly increased leading to a set of very stringent control limits. For practical purposes, a correction factor is needed to be incorporated in these control limits in such situations. Formulation of such correction factors may be an interesting future research problem.

Finally, Kuo and Mittal (1993) cited the following advantages of using control charts from a pragmatic perspective.

1. Skillful application of control charts reduces scrap and rework - which, in turn, reduces production cost and increases productivity.
2. Control charts restrict unnecessary process adjustment by distinguishing between random and assignable causes of variation.
3. Even if all the concerned statistic values are within the stipulated control limits, control charts still provide diagnostic information regarding the stability of a process through the pattern of points plotted in the control chart.

While process capability control charts, in general, inherit all of these advantages of control chart technique; they have some added advantages as listed below.

1. Process capability control charts require lesser sample size as compared to the usual ‘one shot’ estimation of PCIs (refer to Spiring, 1995).
2. Use of process capability control charts makes the sampling scheme more economical in a sense that the sample information already gathered while constructing the corresponding control charts for checking the stability of the process, can be reused here.
3. Process capability control charts enable continuous assessment of a process and hence the common dilemma of production engineers regarding the ideal time for assessing process capability gets resolved.

Since, process capability control charts monitor the consistency of a process in terms of its capability over individual subgroups, they can be used to keep constant vigil on process performance specially for the purpose of tool replacement and supplier selection among others.

Acknowledgment

We are grateful to the anonymous reviewers for helping us to improve the content of this article.

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