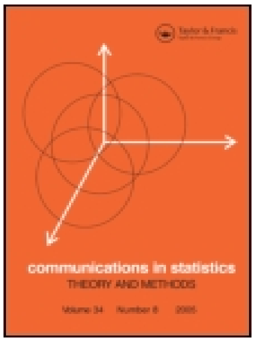

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A Superstructure of Process Capability Indices for Circular Specification Region

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Circular specification regions can be seen in processes like hitting a target (in ballistics), drilling a hole (in manufacturing industries) and so on. However, only a few process capability indices are available in the literature to address the problem. Most of these indices, in turn, make some assumptions like equality of variance and independence of the two axes of the circular tolerance region. Since, in most of the cases, these assumptions are not practically viable, in the present article, we have proposed a few of the process capability indices which do not need the above assumptions to be valid. Also, we propose a superstructure which unifies all the proposed indices. Some properties of these indices have been studied including the threshold value and the relationship of the proportion of non-conformance with the member indices of the superstructure. These strengthen the practical utility of the superstructure. Distributional properties like expectations and variances of the member indices of the superstructure are also studied to have a better insight about the indices. A real life example has been discussed to carry out a comparative study of the performance of the existing as well as the newly developed indices.

Keywords Circular specification region; Elliptical process region; Bi-variate process capability indices; Superstructure; Threshold value; Proportion of non-conformance; Plug-in estimator.

Mathematics Subject Classification 62E15; 62F0; 62P30.

1. Introduction

Process capability indices (PCI's) are used to assess how close a process is to produce what it is supposed to produce. Due to their vast application, these indices are getting more and more importance in practical field, specially in manufacturing industries day by day. Kane (1986) explained the importance of PCI's specially in manufacturing industries. In the literature of process capability indices, computation of most of the indices require two preassigned and distinct specification limits, viz., Upper Specification Limit (USL) and Lower Specification Limit (LSL); see Chan et al. (1988), Boyles (1991), Kotz and Johnson

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(2002), Vannman (1995), and the references there in for more about univariate PCI's with bilateral specification limits. Under the assumption of normality of the distribution of the quality characteristic under consideration, the four classical PCI's for bilateral specification limits are

$$\left.\begin{aligned} C_p &= \frac{USL - LSL}{6\sigma} \\ C_{pk} &= \frac{d - |\mu - M|}{3\sigma} \\ C_{pm} &= \frac{d}{3\sqrt{\sigma^2 + (\mu - T)^2}} \\ C_{pmk} &= \frac{d - |\mu - M|}{3\sqrt{\sigma^2 + (\mu - T)^2}} \end{aligned}\right\}, \tag{1}$$

where USL and LSL are, respectively, the upper and lower specification limits of a process, $d = \frac{USL-LSL}{2}$, $M = \frac{USL+LSL}{2}$ and "T" is the target of the process.

Among these four indices, C_p measures the potential process capability. However, since it only incorporates process standard deviation in its definition, C_{pk} is defined to take into account the process average also. Further, C_{pm} was defined to establish the relationship between squared error loss as far as the target is concerned and the process capability indices. Finally, C_{pmk} was constructed from C_{pk} and C_{pm} to increase the sensitivity (of a PCI) to departure of the process mean "μ" from the target value "T". Vannman (1995) proposed generalized univariate PCI's by developing a superstructure of univariate capability indices called $C_p(u, v)$ which is given by

$$C_p(u, v) = \frac{d - u|\mu - M|}{3\sqrt{\sigma^2 + v(\mu - T)^2}}, u, v \geq 0. \tag{2}$$

Some indices are also available in literature to address the problem of unilateral specification limits, i.e., where either of "USL" and "LSL" of the concerned quality characteristic is available (see Kane, 1986; Chan, et al., 1988; Grau, 2009 and the references therein). Chatterjee and Chakraborty (2012) provided a review of the work done in this area.

Although these two types of specifications are mostly used in manufacturing industries, there are still some parts which remain mostly unattended in literature. Computing capability of a circular specification region is one of such cases. Circularity is the condition of a surface where all points of the surface intersected by any plane perpendicular to a common axis are equidistant from that axis. Following the concept of "Geometric dimensioning and tolerancing" (GD&T), circular specifications can be observed in processes engaging in drilling a hole or from ballistic point of view, hitting a target within a circular region (see Laurent, 1957; Davis et al., 1992). The uniqueness of the processes with circular tolerance region is that the so-called "USL" and/ or "LSL" does not exist corresponding to the concerned quality characteristic and hence classical PCI's are not applicable here. In fact, the following steps are applied to construct the circular (positional) specification region.

1. Identify the target location of the hole or the point to be hit.
2. Treat the location as (0,0) point of co-ordinate geometry and draw the X and Y axes from that point.
3. Draw a circle with center at this (0,0) point and radius as the preassigned tolerable distance from this point. This will give a circular specification region.

Krishnamoorthi (1990) first addressed the problem and proposed two PCI's for circular specifications given by

$$PC_p = \frac{\frac{\pi}{4}D^2}{9\pi\sigma^2} = \frac{1}{36} \times \frac{D^2}{\sigma^2} \tag{3}$$

$$PC_{pk} = \frac{D^2}{4\left[\sqrt{(\overline{X}-a)^2+(\overline{Y}-b)^2}+3\sigma\right]^2}, \tag{4}$$

where D is the diameter of the circular tolerance region; σ is the (equal) standard deviation of the X_1 and X_2 co-ordinates and (a, b) is the target center. When the standard deviation of the X_1 and X_2 co-ordinates, say σ_1 and σ_2, are not equal, $\sigma = \max(\sigma_1, \sigma_2)$. Here, PC_p provides the potential capability of a process and PC_{pk} gives its actual capability. Note that, Krishnamoorthi's (1990) PCI's (though unit-less) consider the ratio of two areas which are expressed in squared units. Hence, to make them comparable to conventional PCI's square root of these indices should be considered. Also, the index PC_{pk}, though named analogous to C_{pk} of bilateral specifications, works actually like C_{pm} in a sense that the numerator is the same as that of PC_p while the denominator is modified to measure the proximity of the process mean to the target.

Davis et al. (1992) formulated the PCI of a process with circular specification as a function of the minimum fraction non-conforming produced by the process. For this they considered the Euclidian distance between the target position and the actual position of the center of the feature as the measurable quality characteristic and thus ignored the underlying correlation between the two axes of the specification region.

Bothe (2006) applied control chart technique to measure the stability of the quality characteristic corresponding to each of the two axes as well as the radial distances of the observed center locations from the target and proposed the following PCI's for circular tolerance region:

$$\left.\begin{aligned}
\widehat{C}_P &= \frac{USL-\widehat{\mu_C}}{3\widehat{\sigma}_{ST,C}}\\
\widehat{P}_P &= \frac{USL-\widehat{\mu_C}}{3\widehat{\sigma}_{LT,C}}\\
\widehat{C}_{PK} &= \frac{USL-\widehat{\mu}_r}{3\widehat{\sigma}_{ST}}\\
\widehat{P}_{PK} &= \frac{USL-\widehat{\mu}_r}{3\widehat{\sigma}_{LT}}
\end{aligned}\right\} \tag{5}$$

where $\widehat{\mu_C} = \frac{\sum_{i=1}^{n} r_{C,i}}{n}$; $r_{C,i} = \sqrt{(x_i-\overline{x})^2+(y_i-\overline{y})^2}$; "n" is the sample size; $\overline{MR} = \frac{\sum_{i=2}^{n} MR_i}{n-1}$, MR_i's are obtained from moving range chart; $\widehat{\sigma_{ST}} = \frac{\overline{MR}}{1.128}$; $\widehat{\sigma_{LT}} = \frac{1}{c_4}\sqrt{\frac{\sum_{i=1}^{n}(r_i-\overline{r})^2}{n-1}}$, with, $r_i = \sqrt{x_i^2+y_i^2}$; $\widehat{\mu}_r = \frac{\sum_{i=1}^{n} r_i}{n}$; $\widehat{\sigma}_{ST,C} = \frac{\overline{MR_C}}{d_2}$, where MR_C values are obtained from the moving range chart of the data set after the target hole location is shifted to the middle of the cluster of actual hole centers and d_2 is a function of the sample size "n" and $\widehat{\sigma}_{LT,C} = \frac{1}{c_4}\sqrt{\frac{\sum_{i=1}^{n}(r_{C,i}-\overline{r_C})^2}{n-1}}$ and c_4 is a constant based on the sample size "n".

The commonality between all of these indices is that they are based on the following two assumptions (see Davis et al., 1992):

1. The variation of the quality characteristic along the two axes are the same.
2. The random variables corresponding to two axes are mutually independent.

However, due to several practical reasons the assumptions of homoscedasticity and independence of the two axes may be violated (Chew and Boyce, 1962). This gives an elliptical shape to the process region. Moreover, for normally distributed characteristics, even under the said assumptions, the underlying distribution of the radial distance between the actual process center and the target center will no longer remain normal—rather, it will follow radial error distribution. Hence while measuring capability of processes which are characterized by the radial distance as the measurable characteristic, using C_{pk}, C_{pm}, or other PCI's specially designed for normally distributed process characteristics (the approach considered by Bothe, 2006), may often yield inconsistent as well as misleading results.

In this context, following Karl et al. (1994), the problem of circular specification may also be designed as a multivariate (specifically, bi-variate) process capability problem with X_1 and X_2 being the two correlated variables along the two axes of the specification region. Note that in the analysis of radial error for two dimensional case, the X_1 component is the azimuth or deflection error, and the X_2 component is the range or pitch error. Azimuth and range errors are associated with ground or horizontal targets, and deflection and pitch errors are associated with vertical targets (Culpepper, 1978). In the present paper, we have adopted this approach to propose a superstructure of PCI's called $C_{p,c}(u, v)$ dealing with circular specification region and having more general field of application in a sense that $C_{p,c}(u, v)$ does not take into account the assumptions of independence and homoscedasticity of the two axes.

We discuss various indices belonging to $C_{p,c}(u, v)$ in the next section. Section 3 deals with some crucial properties of $C_{p,c}(u, v)$ including its relationship with proportion of non-conformance and threshold value. The discussion regarding the plug-in estimator of $C_{p,c}(u, v)$ and its expectation and variance is in Sec. 4 followed by a comparison between the performance of $C_{p,C}(u, v)$ and existing PCI's in this field based on some real life as well as simulated data sets in Sec. 5. Section 6 deals with an application of $C_{p,c}(u, v)$ in a real life data set. Finally, we conclude the article in Sec. 7 with a brief summary.

2. New Super-structure of Process Capability Indices for Processes with Positional (Circular) Tolerances

Here we assume the underlying distribution of the quality characteristics to be bivariate normal, i.e., if X_1 and X_2 are the two directional process variables, then

$$\boldsymbol{X} = (X_1, X_2) \sim N_2\left(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho = \frac{\sigma_{12}}{\sigma_1\sigma_2}\right).$$

As we have already mentioned, for practical purposes the "target center" of the hole is considered to be the (0,0) point of the co-ordinate axes. Moreover, the tolerance region should measure the allowable deviation from the "target center" instead of the process average (mean vector). As such, if the diameter of the circular tolerance region is 'D' units and if the center of the elliptical process region coincides with that of the circular tolerance region, i.e., (0,0) point, then the potential capability of a process to manufacture products within the pre-assigned tolerance may be expressed as

$$C_{p,c} = \sqrt{\frac{\text{Area of a circular specification region with diameter 'D' units}}{\text{Area of a } 100(1-\alpha)\% \text{ constant contour elliptical process region}}} \tag{6}$$

The numerator of Eq. (6) can be computed as $\sqrt{\pi(\frac{D}{2})^2}$ units, i.e., $\sqrt{\frac{\pi}{4}D^2}$ units while the value of the denominator will be $\sqrt{\chi^2_{\alpha,2}\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$ units, where, $\chi^2_{\alpha,2}$ represents the value of a χ^2 distribution with "2" d.f. that has a right tail area of "α" units. Hence, $C_{p,c}$ can be re-written as

$$\begin{aligned} C_{p,c} &= \sqrt{\frac{\frac{\pi}{4}D^2}{\chi^2_{\alpha,2}\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}} \\ &= \frac{D}{2\sqrt{\chi^2_{\alpha,2}\sigma_1\sigma_2\sqrt{1-\rho^2}}} \\ &= \frac{D}{2\sqrt{\chi^2_{\alpha,2}\sqrt{|\Sigma|}}}. \end{aligned} \tag{7}$$

For example, since for $\alpha = 0.01$, $\chi^2_{0.01,2} = 9.210$, hence for the 99% constant contour ellipsoid, Eq. (7) can be written as

$$C_{p,c} = \frac{D}{6.07\sqrt{\sigma_1\sigma_2\sqrt{1-\rho^2}}}. \tag{8}$$

Note that $C_{p,C}$ incorporates the correlation structure between the two variables and also does not require them to have equal variances. However this index, similar to C_p of bi-lateral specification limits, does not take into account the process centering and hence determines potential process capability instead of indicating actual capability of the process. Hence, to measure the actual capability of a process, we have to define indices similar to C_{pk}, C_{pm}, and so on. For this reason, let us assume that the centers of the circular specification region and the process region are $(0, 0)'$ and $(\mu_1, \mu_2)'$, respectively. According to Bothe (2006), "a hole center is considered to be within specification, when its radial distance from the target location is less than the radius of the true position circle." The author used average radial distance of the observations from the target center as the measure of location.

Following Bothe's (2006) approach, the actual center of the process region can be considered as the $\boldsymbol{\mu} = (\mu_1, \mu_2)'$, where μ_1 and μ_2 are, respectively, the arithmetic mean of the X_1-axis and X_2-axis values of the measured quality characteristics. Hence, the radial distance between the target center and the actual process center will be $\sqrt{\boldsymbol{\mu}'\boldsymbol{\mu}} = \sqrt{\mu_1^2 + \mu_2^2}$ units. Analogous to C_{pk} of bilateral specification limits, we can therefore define a PCI for circular specification as

$$C^*_{pk,c} = C_{p,c} - \frac{\sqrt{\boldsymbol{\mu}'\boldsymbol{\mu}}}{\sqrt{\pi\chi^2_{\alpha,2}\sigma_1\sigma_2\sqrt{1-\rho^2}}}$$

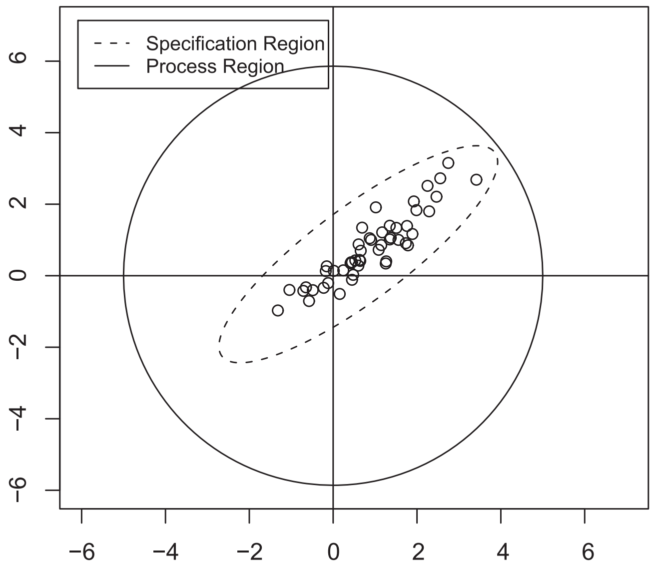

Figure 1. Process with all the observations within the process region.

$$= \frac{\frac{D}{2} - \sqrt{\frac{\boldsymbol{\mu}'\boldsymbol{\mu}}{\pi}}}{\sqrt{\chi^2_{\alpha,2}\sigma_1\sigma_2\sqrt{1-\rho^2}}}. \tag{9}$$

However, $C^*_{pk,c}$ may not always be able to capture the actual capability of a process. This can be explained through the following two situations. Here for both the cases we consider the circular specification region to have center at (0, 0) point with the diameter equal to 7 units. Also, we draw 50 random samples from two different processes both of them following bi-variate normal distribution with mean vector as $\boldsymbol{\mu} = (0.6, 0.6)'$ and dispersion matrix $\begin{pmatrix} 1.2 & 0.93 \\ 0.93 & 1.0 \end{pmatrix}$ i.e., the correlation coefficient is 0.85.

Now, from Fig. 1, we can observe that, for the first process, all the sample observations are within the specification region. On the contrary, as can be seen in Fig. 2, 2 of the 50 observations lie out side of the modified process region. Hence the first process is more capable than the second one. However, $C^*_{pk,c}$ will consider both the processes as equally capable.

Here the problem lies in the fact that the radial distance of the process center from the specification center has been used as the measure of deviation from ideal location, which smooths out the impact of the radial distance of individual observations from the process center. This can be addressed by replacing $\boldsymbol{\mu}'\boldsymbol{\mu}$ in Eq. (9) by $\mu^* = \frac{1}{n}\sum_{i=1}^{n} d_i^*$, where, $d_i^* = \sqrt{(X_{1_i} - \mu_1)^2 + (X_{2_i} - \mu_2)^2} = \sqrt{(\boldsymbol{X}_i - \boldsymbol{\mu})'(\boldsymbol{X}_i - \boldsymbol{\mu})}$, where, $\boldsymbol{X}_i = (X_{1i}, X_{2i})'$ for $i = 1, 2, \cdots, n$ and "n" is the number of sample observations. Thus, we can redefine $C^*_{pk,c}$ as

$$C_{pk,c} = \frac{\frac{D}{2} - \frac{\mu^*}{\sqrt{\pi}}}{\sqrt{\chi^2_{\alpha,2}\sigma_1\sigma_2\sqrt{1-\rho^2}}} \tag{10}$$

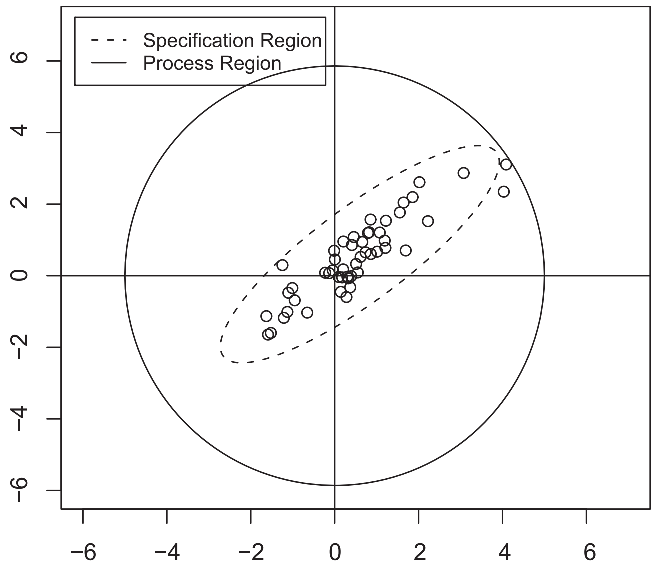

Figure 2. Process with some of the observations outside the process region.

Here, we have used Euclidian distance instead of Mahalanobis distance as otherwise it will become unit free and hence its subtraction from $\frac{D}{2}$ in (10) will no longer be possible. Note that for $\boldsymbol{\mu} = \mathbf{0}$, $\boldsymbol{\mu}^*$ gives the average distance of the observed data points from the target center and hence for a stable process, it should be less than the radius ($\frac{D}{2}$ units). This also ensures the non-negative value of $C_{pk,c}$ for a stable process and this characteristics of $C_{pk,c}$ is similar to the non-negativity assumption of C_{pk} of symmetric bilateral specifications.

Boyles (1991) thoroughly discussed the advantages of C_{pm} over C_{pk} for bilateral specification limits, especially while computing the proximity of process centering to the target. As has been discussed earlier although Krishnamoorthi (1990) proposed PC_{pk} analogous to C_{pm} for circular specification, his index does not take into account the possible differences in the variances along the two axes as well as the correlation between them. Taking these two draw backs of PC_{pk} into account, we can construct PCI for circular specification region, analogous to C_{pm}, in two different ways viz: (i) following Krishnamoorthi's (1990) approach and (ii) direct method.

Now, for the first case, the numerator of the index will be same as that of $C_{p,c}$ in (7) and in the denominator, we have to incorporate the contribution of the proximity of the process mean to the target center measured by $\boldsymbol{\mu}'\boldsymbol{\mu}$ (Euclidian distance) or $\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}$ (Mahalanobis distance). Thus, we have

$$C^*_{pm,c} = \frac{D\sqrt{\pi}}{2\left[\sqrt{\pi\chi^2_{\alpha,2}\sigma_1\sigma_2\sqrt{1-\rho^2}} + \sqrt{\boldsymbol{\mu}'\boldsymbol{\mu}}\right]} \tag{11}$$

Again, C_{pm} in (1) can be written as $C_{pm} = C_p \times \frac{1}{\sqrt{1+(\frac{\mu-T}{\sigma})^2}}$, where "T" is the target value of the characteristic. Thus, for circular tolerance region, we can define using direct method,

$$C_{pm,c} = C_{p,c} \times \frac{1}{\sqrt{1+\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}}}$$

$$= \frac{D}{2\sqrt{\chi^2_{\alpha,2}\sigma_1\sigma_2\sqrt{1-\rho^2}}} \times \frac{1}{\sqrt{1+\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}}} \tag{12}$$

Next, analogous to C_{pmk} of bilateral specification limits, we can define $C_{pmk,c}$ by combining (10) and (12) as

$$C_{pmk,c} = \frac{\frac{D}{2} - \frac{\boldsymbol{\mu}^*}{\sqrt{\pi}}}{\sqrt{\chi^2_{\alpha,2}\sigma_1\sigma_2\sqrt{1-\rho^2}}} \times \frac{1}{\sqrt{1+\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}}} \tag{13}$$

Finally, similar to $C_p(u, v)$ of (2) proposed by Vannman (1995), a superstructure of PCI's for circular specification region may be defined as

$$C_{p,c}(u, v) = \frac{\frac{D}{2} - \frac{u}{\sqrt{\pi}}\mu^*}{\sqrt{\chi^2_{\alpha,2}\sigma_1\sigma_2\sqrt{1-\rho^2}}} \times \frac{1}{1+v\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}}, \tag{14}$$

where "u" and "v" are two nonnegative parameters. Here, $C_{p,c}(0, 0) = C_{p,c}$, $C_{p,c}(1, 0) = C_{pk,c}$, $C_{p,c}(0, 1) = C_{pm,c}$ and $C_{p,c}(1, 1) = C_{pmk,c}$. Following are some salient features of $C_{p,c}(u, v)$:

1.

$$C_{p,c}(u, v) \leq C_{p,c}(u, 0) \leq C_{p,c}(0, 0)$$
$$C_{p,c}(u, v) \leq C_{p,c}(0, v) \leq C_{p,c}(0, 0), \forall u \geq 0, v \geq 0$$

There is no clear-cut relationship between $C_{pk,c}$ and $C_{pmk,c}$. This interrelationship between the member PCI's of $C_{p,c}(u, v)$ are similar to the findings of Kotz and Johnson (2002) for C_p, C_{pk}, C_{pm}, and C_{pmk} for the univariate case. Also,

$$\begin{aligned} C_{pm,c} &> C^*_{pm,c}, \quad \text{if} \quad \sqrt{\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} > \sqrt{\frac{\boldsymbol{\mu}'\boldsymbol{\mu}}{\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}}} \\ &= C^*_{pm,c}, \quad \text{if} \quad \sqrt{\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} = \sqrt{\frac{\boldsymbol{\mu}'\boldsymbol{\mu}}{\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}}} \\ &< C^*_{pm,c}, \quad \text{if} \quad \sqrt{\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} < \sqrt{\frac{\boldsymbol{\mu}'\boldsymbol{\mu}}{\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}}} \end{aligned}$$

2. For any fixed correlation coefficient, all the indices increase if either or both of σ_1 and σ_2 decrease. A desirable property for any good process is that its variance should be as minimum as possible.
3. For fixed σ_1 and σ_2, the indices increase with the increase in the correlation (ρ) between the X_1 and X_2 axes.
4. $C_{pm,c}$ is more user-friendly than $C^*_{pm,c}$ in a sense that it provides some measure of the degree of correction required for proper process centering. In fact the value of $\frac{1}{\sqrt{1+\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}}}$ should ideally be one which is attained if $\boldsymbol{\mu} = \boldsymbol{T} = \mathbf{0}$, i.e., if process centering coincides with the target center. The lower the value of this constant, the higher will be the degree of

departure of average process centering from target. As such consideration of $\frac{1}{\sqrt{1+\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}}}$ along with $C_{pm,c}$ depicts the true picture of a process.

5. $C_{p,c}(u, v)$ is optimum on target.

3. Some More Interesting Properties of $C_{p,c}(u, v)$

In this section, two very crucial properties of $C_{p,c}(u, v)$ are discussed viz., threshold value and the relationship with the proportion of non-conformance. Both of these properties are discussed with respect to $C_{p,c}$, measuring potential capability of a process with circular process region.

3.1 Threshold Value of $C_{p,c}$

The concept of threshold value is very important specially from the view point of the interpretation of the computed index value. The threshold value signifies such a level of a process beyond which it is considered to be capable of producing which it is supposed to produce, while an index value below the threshold gives clear indication of the unsatisfactory performance of the process. In this context, since $C_{p,c}$ measures the potential capability of a process, conventionally, threshold value is computed only for this PCI. The logic behind this is that if a process is not even potentially capable, there is no point in carrying out further computation regarding its actual capability. In practice, often the threshold value of $C_{p,c}$ is considered as the threshold value of the corresponding PCI's measuring actual capability of a process.

Note that for the case of bilateral specifications (eg., C_P), generally, the threshold value is "1" which is attained at that level of the process where the process region coincides with the specification region. Although in our case this basic approach will remain same, we have to take into account another aspect of the process, i.e., the correlation coefficient (ρ) between the two variables of interest.

Like C_p, for which the threshold value is reached at $USL - LSL = 6\sigma$, for $C_{p,c}$ with nonzero correlation coefficient among the two variables, the maximum variance for either variable will be $\frac{D}{2}$ units for just capable scenario as otherwise some part of the process region will lie outside the modified process region. Let $\sigma_{\max} = \max(\sigma_1, \sigma_2) = \frac{D}{2}$ and $\sigma_{\min} = \min(\sigma_1, \sigma_2)$. Then, from (7), the threshold value, $C^T_{p,c}$ of $C_{p,c}$ will be

$$C^T_{p,c} = \frac{\frac{D}{2}}{\sqrt{\chi^2_{\alpha,2} \times \frac{D}{2} \times \sigma_{\min}\sqrt{1-\rho^2}}} = \sqrt{\frac{D}{2 \times \chi^2_{\alpha,2} \times \sigma_{\min} \times \sqrt{1-\rho^2}}}. \tag{15}$$

Note that for $\rho = 0$ and $\sigma_1 = \sigma_2 = \frac{D}{2}$, i.e., when the two variables of the elliptical process region are uncorrelated and have equal variances, then the process region coincides with the specification region and hence following the conventional approach of computing threshold value of PCI's here also the threshold value should be "1". However as can be seen from (15), this is not the case here. The reason behind this is that we considered the process region to be a $100(1-\alpha)\%$ constant contour ellipsoid and this will contribute to the expression of the threshold value of $C_{p,c}$ even when the other factors like correlation and unequal variance are nullified or neutralized. As such for $\rho = 0$ and $\sigma_1 = \sigma_2 = \frac{D}{2}$, the threshold value of $C_{p,c}$ will be $C^{T*}_{p,c} = \sqrt{\frac{1}{\chi^2_{\alpha,2}}} = \frac{1}{\chi_{\alpha,2}}$ and hence $C_{p,c}$ and in general $C_{p,c}(u, v)$ is not very suitable when there is zero or very low correlation between the two variables.

Finally, the threshold value of $C_{p,c}$ can be considered as the threshold value of $C_{pm,c}$ as well since for $\boldsymbol{\mu} = \mathbf{0}$, $C_{pm,c}$ boils down to $C_{p,c}$. Even when $C_{p,c} \geq C^T_{p,c}$ but $C_{pm,c} < C^T_{p,c}$, the process is likely to be off-target. On the other hand, if $C_{pm,c} > C^T_{p,c}$, the process can be considered to be capable. Since $C_{pmk,c} \leq C_{pm,c} \leq C_{p,c}$, such observation is true for $C_{pmk,c}$ as well.

Since $C_{pk,c}$ does not boil down to $C_{p,c}$ for $\boldsymbol{\mu} = \mathbf{0}$, similar conclusion can not be drawn for this PCI. However this is not a serious drawback as the $C_{pk,c}$ values should not be directly used to judge the capability level of a process as that may give misleading result. Rather, it is useful for computing the expected proportion of non-conforming items produced by the process—the expression for which is developed in Sec. 3.2.

3.2 Relationship of Proportion of NonConformance with $C_{p,c}$ and $C_{pk,c}$

A process with higher value of PCI is supposed to produce lower proportion of non-conforming items. Hence a good PCI should be a one-to-one function of such proportion. Following the same logic as that for the threshold value, usually, statistical relationship is established between the potential capability of a process and the proportion of nonconformance.

Now, let us first consider the position of an actual observation with respect to the circular specification region as has been depicted in Fig. 3.

From this figure, with the center at the (0, 0) point, the general expression for the proportion of nonconformance "P_{NC}" can be formulated as follows:

$$P_{NC} = P\left[(\boldsymbol{X} - \mathbf{0})'\Sigma^{-1}(\boldsymbol{X} - \mathbf{0}) > \begin{pmatrix} \frac{D}{2} & 0 \end{pmatrix} I_2 \begin{pmatrix} \frac{D}{2} \\ 0 \end{pmatrix}\right], \tag{16}$$

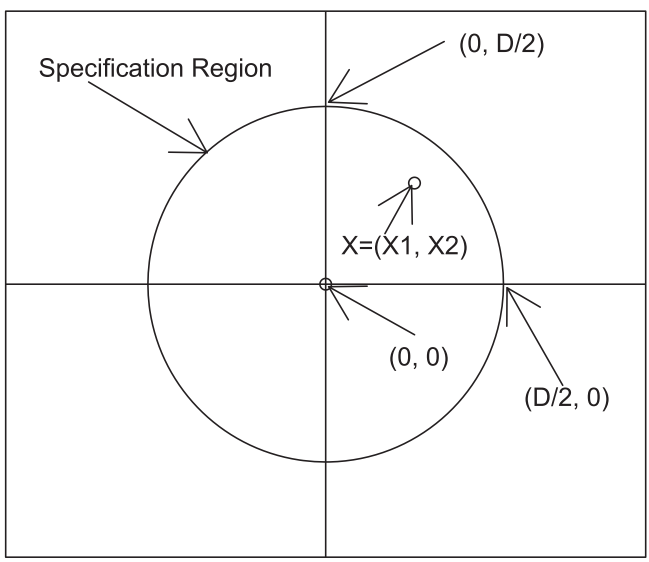

Figure 3. Position of an observed point with respect to the circular specification region.

where "I_2" is a (2×2) identity matrix. Assuming that the process is centered on target, P_{NC} can be written as

$$\begin{aligned}
P_{NC} &= P\left[\boldsymbol{X}'\Sigma^{-1}\boldsymbol{X} > \begin{pmatrix}\frac{D}{2} & 0\end{pmatrix} I_2 \begin{pmatrix}\frac{D}{2} \\ 0\end{pmatrix} | \boldsymbol{X} \sim N_2(\boldsymbol{0}, \Sigma)\right] \\
&= P\left[\chi^2_{\alpha,2} > \frac{D^2}{4}\right] \\
&= P\left[\frac{D^2}{4\chi^2_{\alpha,2}\sigma_1\sigma_2\sqrt{1-\rho^2}} < \frac{1}{\sqrt{|\Sigma|}}\right] \\
&= P\left[C^2_{p,c} < \frac{1}{\sqrt{|\Sigma|}}\right] \\
&= P\left[\sqrt{|A|}C^2_{p,c} < \sqrt{\chi^2_{\alpha,n-1} \times \chi^2_{\alpha,n-2}}\right], \text{ since } \frac{|A|}{|\Sigma|} \sim \prod_{i=1}^{p} \chi^2_{n-i} \\
&= P\left[\sqrt{\chi^2_{\alpha,n-1} \times \chi^2_{\alpha,n-2}} > (n-1)\sqrt{|S|}C^2_{p,c}\right] \\
&= P\left[\sqrt{\left\{\frac{\chi^2_{\alpha,2n-4}}{2}\right\}^2} > (n-1)\sqrt{|S|}C^2_{p,c}\right], \text{ from Pearn et al. (2007)} \\
&= P\left[\chi^2_{\alpha,2n-4} > 2(n-1)\sqrt{|S|}C^2_{p,c}\right] \qquad (17)
\end{aligned}$$

where $A = (n-1)S$ with "S" being the sample variance-covariance matrix and "n" is the sample size.

As such, if either of P_{NC} or $C_{p,c}$ is known, the other can be readily obtained. However, since, when the process is not on target, $C_{p,c}$ measures the potential capability of a process instead of the actual capability, P_{NC} gives the minimum observable proportion of nonconformance for $\boldsymbol{\mu} \neq \boldsymbol{0}$. It is required to derive the expression for the expected proportion of nonconformance for such processes and let us denote this proportion as P^E_{NC}. Then,

$$\begin{aligned}
P^E_{NC} &= P\left[\boldsymbol{X}'\Sigma^{-1}\boldsymbol{X} > \frac{D^2}{4} | \boldsymbol{X} \sim N_2(\boldsymbol{\mu}, \Sigma)\right] \\
&= P\left[\frac{D^2}{4} < \chi^2_{\alpha,2}(\lambda)\right] \\
&= P\left[C_{pk,c} < \left\{\sqrt{\frac{\chi^2_{\alpha,2}(\lambda)}{\chi^2_{\alpha,2}}} - \frac{\boldsymbol{\mu}^*}{\sqrt{\pi} \times \sqrt{\chi^2_{\alpha,2}}}\right\} \times \frac{1}{|\Sigma|^{\frac{1}{4}}}\right] \\
&= P\left[2(n-1)\sqrt{|S|}C^2_{pk,c} < \left\{\sqrt{F_{\alpha,2,2}(\lambda)} - \frac{\boldsymbol{\mu}^*}{\sqrt{\pi}} \times \frac{1}{\sqrt{\chi^2_{\alpha,2}}}\right\}^2 \times \sqrt{\left(\chi^2_{\alpha,2n-4}\right)^2}\right] \\
&= P\left[2(n-1)\sqrt{|S|}C^2_{pk,c} < \sqrt{F_{\alpha,2,2}(\lambda) \times \chi^2_{\alpha,2n-4}} - \frac{\boldsymbol{\mu}^*}{\sqrt{\pi}} \times \sqrt{F_{\alpha,2n-4,2}}\right] \qquad (18)
\end{aligned}$$

Since $C_{pk,c}$ measures the actual capability of a process, P_{NC}^{E}, being a function of $C_{pk,c}$, measures the expected proportion of non-conformance. However, exact relationship between process non-conformance and $C_{pm,c}$ or $C_{pmk,c}$ does not exist as has been the case for C_{pm} and C_{pmk} observed by Vannman (1995). It may be noted that P_{NC} can be expressed as a lower bound for some probabilities expressed as functions of $C_{pm,c}$ and $C_{pmk,c}$.

$$P_{NC} \leq P\left[2(n-1)\sqrt{|S|}C_{pm,c}^{2} < I_{F}^{*} \times \chi_{\alpha,2n-4}^{2}\right] \tag{19}$$

$$P_{NC} \leq P\left[2(n-1)\sqrt{|S|}C_{pmk,c}^{2} < I_{F}^{*} \times \chi_{\alpha,2n-4}^{2}\right], \tag{20}$$

where,

$$I_{F}^{*} = \begin{cases} 1 & \text{if } \boldsymbol{X} \sim N_2(\mathbf{0}, \Sigma) \\ F_{\alpha,2,2}(\delta) & \text{if } \boldsymbol{X} \sim N_2(\boldsymbol{\mu}, \Sigma) \end{cases}$$

.

4. Plug-in Estimator of $C_{p,c}(u, v)$ and its Expectation and Variance

Since in most of the practical situations, the values of the process parameters remain unknown, one has to depend upon the PCI values computed on the basis of the sample information. For $C_{pc}(u, v)$, the corresponding plug-in estimator will be

$$\begin{aligned}\widehat{C}_{p,c}(u, v) &= \frac{\frac{D}{2} - \frac{u}{\sqrt{\pi}}\widehat{\mu^*}}{\sqrt{\chi_{\alpha,2}^{2} s_1 s_2 \sqrt{1-r^2}}} \times \frac{1}{1 + v\overline{\boldsymbol{X}}' S^{-1}\overline{\boldsymbol{X}}} \\ &= \frac{\frac{D}{2} - \frac{u}{\sqrt{\pi}}\widehat{\mu^*}}{\sqrt{\chi_{\alpha,2}^{2}\sqrt{|S|}}} \times \frac{1}{1 + v\overline{\boldsymbol{X}}' S^{-1}\overline{\boldsymbol{X}}},\end{aligned} \tag{21}$$

where $\widehat{\mu^*} = \frac{1}{n}\sum_{i=1}^{n}\widehat{d_i^*}$, $\widehat{d_i^*} = \sqrt{(X_{1_i} - \overline{X_1})^2 + (X_{2_i} - \overline{X_2})^2}$, $i = 1, 2, \cdots, n$, $\overline{\boldsymbol{X}} = (\overline{X_1}, \overline{X_2})'$ with $\overline{X_i}$ being the sample mean corresponding to X_i axis, for $i = 1, 2$; s_1 and s_2 are the sample standard deviations corresponding to the variables "X_1" and "X_2" respectively; "r" is the sample correlation coefficient between the two variables and "S" is the corresponding sample variance-covariance matrix such that $S = \begin{pmatrix} s_1^2 & s_{12} \\ s_{12} & s_2^2 \end{pmatrix}$ with $s_{12} = r s_1 s_2$.

However, due to sampling fluctuations, mere computation of such PCI value using (21) may not, always, reveal the actual capability of the concerned process and hence the characteristics of this plug-in estimator, especially its expectation and variance, needs to be studied. For this, we first compute $E[\widehat{C}_{p,c}(0, 0)]$. From (21), the expression for $\widehat{C}_{p,c}(0, 0)$ will be

$$\begin{aligned}\widehat{C}_{p,c}(0, 0) &= \frac{D}{2\sqrt{\chi_{\alpha,2}^{2} s_1 s_2 \sqrt{1-r^2}}} \\ &= \frac{D}{2\sqrt{\chi_{\alpha,2}^{2}}} \times |S|^{-\frac{1}{4}}, \text{ since, } |S|^{\frac{1}{2}} = s_1 s_2\sqrt{1-r^2}.\end{aligned} \tag{22}$$

Now, if $|S|$ is the determinant of the sample variance-covariance matrix (popularly known as sample generalized variance), then for $p = 2$, $|S| \sim \frac{|\Sigma|}{(n-1)^2} \prod_{i=1}^{2} \chi^2_{n-i}$ (Giri, 2004). Hence,

$$
\begin{aligned}
E[\widehat{C}_{p,c}(0,0)] &= \frac{D}{2\sqrt{\chi^2_{\alpha,2}}} \times E\left[|S|^{-\frac{1}{4}}\right] \\
&= \left\{ \frac{\sqrt{n-1}\,\Gamma(\frac{2n-5}{2})}{\Gamma(n-2)} \right\} \times C_{p,c}(0,0) \\
&= k(n) \times C_{p,c}(0,0), \text{ say, where, } k(n) = \frac{\sqrt{n-1}\,\Gamma(\frac{2n-5}{2})}{\Gamma(n-2)}
\end{aligned}
$$

$$
\Rightarrow E\left[\frac{1}{k(n)} \widehat{C}_{p,c}(0,0)\right] = C_{p,c}(0,0) \tag{23}
$$

Therefore, $\widehat{C}^*_{p,c}(0,0) = \frac{1}{k(n)} \widehat{C}_{p,c}(0,0)$ is an unbiased estimator of $C_{p,c}(0,0)$.

Also,

$$
\begin{aligned}
E\left[\widehat{C}^2_{p,c}(0,0)\right] &= \frac{D^2}{4\chi^2_{\alpha,2}} \times E[|S|^{-\frac{1}{2}}] \\
&= \left[C^2_{p,c}(0,0)\right] \times \frac{n-1}{n-3}
\end{aligned} \tag{24}
$$

and hence

$$
V[\widehat{C}_{p,c}(0,0)] = (n-1) \times \left\{ \frac{1}{n-3} - \left[\frac{\Gamma(\frac{2n-5}{2})}{\Gamma(n-2)} \right]^2 \right\} \times C^2_{p,c}(0,0). \tag{25}
$$

Now, from (21), the expression for $\widehat{C}_{p,c}(u,0)$ will be

$$
\begin{aligned}
\widehat{C}_{p,c}(u,0) &= \frac{\frac{D}{2} - \frac{u}{\sqrt{\pi}} \widehat{\mu^*}}{\sqrt{\chi^2_{\alpha,2}\sqrt{|S|}}} \\
&= \widehat{C}^*_{p,c}(0,0) - u \times \frac{\widehat{\mu^*} \times |S|^{-\frac{1}{4}}}{\sqrt{\pi\chi^2_{\alpha,2}}} \\
\Rightarrow E[\widehat{C}_{p,c}(u,0)] &= E[\widehat{C}_{p,c}(0,0)] - \frac{u}{\sqrt{\pi\chi^2_{\alpha,2}}} \times E[\widehat{\mu^*} \times |S|^{-\frac{1}{4}}] \\
&= k(n) \times C_{p,c}(0,0) - \frac{u}{\sqrt{\pi\chi^2_{\alpha,2}}} \times E[\widehat{\mu^*} \times |S|^{-\frac{1}{4}}],
\end{aligned} \tag{26}
$$

Note that since $\widehat{\mu^*}$ and $|S|^{-\frac{1}{4}}$ are functions of "$\widehat{\boldsymbol{\mu}}$" and "S" respectively, they are also mutually independent. Hence, $E[\widehat{\mu^*} \times |S|^{-\frac{1}{4}}] = E[\widehat{\mu^*}] \times E[|S|^{-\frac{1}{4}}]$. Now,

$$
E[|S|^{-\frac{1}{4}}] = \frac{\sqrt{(n-1)}}{|\Sigma|^{\frac{1}{4}}} \times \left[\frac{\Gamma(\frac{2n-5}{2})}{\Gamma(n-2)} \right]. \tag{27}
$$

Moreover, if $(Y_1, Y_2)' \sim N_2(0, 0, \sigma_1^2, \sigma_2^2, \rho = \frac{\sigma_{12}}{\sigma_1\sigma_2})$, then for the radial error $\sqrt{Y_1^2 + Y_2^2}$, expected value will be $E[\sqrt{Y_1^2 + Y_2^2}] = \sqrt{\frac{2}{\pi}}\Im(K)\sigma_1^{*2}$ and $E[Y_1^2 + Y_2^2] = (2 - K^2)\sigma_1^{*2}$ (Scheur, 1962), where:

- $\sigma_1^{*2} = \frac{1}{2}\left\{(\sigma_1^2 + \sigma_2^2) + \left[(\sigma_1^2 - \sigma_2^2)^2 + 4\sigma_{12}^2\right]^{\frac{1}{2}}\right\}$;
- $\sigma_2^{*2} = \frac{1}{2}\left\{(\sigma_1^2 + \sigma_2^2) - \left[(\sigma_1^2 - \sigma_2^2)^2 + 4\sigma_{12}^2\right]^{\frac{1}{2}}\right\}$;
- $K^2 = \frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2}$;
- $\Im(K)$ is the complete elliptical integral of the 2^{nd} kind (Legendre, 1932).

Hence, $E[\widehat{\mu^*}] = \sqrt{\frac{2}{\pi}}\Im(K)\sigma_1^{*2}$ and as such, from (26),

$$E[\widehat{C}_{p,c}(u,0)] = \sqrt{\frac{n-1}{\chi^2_{\alpha,2}\sqrt{|\Sigma|}}} \times \left[\frac{D}{2} - \frac{u\sqrt{2}}{\pi} \times \Im(K)\sigma_1^*\right] \times \frac{\Gamma(\frac{2n-5}{2})}{\Gamma(n-2)}. \tag{28}$$

Also,

$$\begin{aligned} E\left[\widehat{C}^2_{p,c}(u,0)\right] &= \frac{1}{\chi^2_{\alpha,2}} \times E\left\{\left[\frac{D}{2} - \frac{u\sqrt{2}}{\pi}\widehat{\mu^*}\right]^2\right\} \times E\left(|S|^{-\frac{1}{2}}\right) \\ &= \frac{1}{\chi^2_{\alpha,2}} \times \left\{\frac{D^2}{4} + \frac{u}{\pi}\sigma_1^* \times \left[u(2-k^2)\sigma_1^* - D\sqrt{2}\Im(k)\right]\right\} \frac{n-1}{(n-3)\sqrt{|\Sigma|}}. \end{aligned} \tag{29}$$

Hence, from (28)and (29),

$$\begin{aligned} V[\widehat{C}_{p,c}(u,0)] = \frac{n-1}{\chi^2_{\alpha,2}\sqrt{|\Sigma|}} &\times \left[\frac{1}{n-3} \times \left[\frac{D^2}{4} + \frac{u}{\pi}\sigma_1^* \times \left\{u(2-k^2)\sigma_1^* - D\sqrt{2}\Im(k)\right\}\right]\right. \\ &- \left.\left\{\frac{D}{2} - \frac{u\sqrt{2}}{\pi} \times \Im(K)\sigma_1^*\right\}^2 \times \left(\frac{\Gamma(\frac{2n-5}{2})}{\Gamma(n-2)}\right)^2\right]. \end{aligned} \tag{30}$$

Again, from (21), for $\widehat{C}_{p,c}(0, v)$, we have

$$\begin{aligned} E[\widehat{C}_{p,c}(0,v)] &= E\left[\widehat{C}_{p,c}(0,0) \times \left(1 + v\overline{\boldsymbol{X}}'S^{-1}\overline{\boldsymbol{X}}\right)^{-\frac{1}{2}}\right] \\ &= E\left[\widehat{C}_{p,c}(0,0)\right] \times E\left[\left(1 + v\overline{\boldsymbol{X}}'S^{-1}\overline{\boldsymbol{X}}\right)^{-\frac{1}{2}}\right] \\ &= k(n)C_{p,c}(0,0) \times \left\{1 - \frac{v}{2}E(\overline{\boldsymbol{X}}'S^{-1}\overline{\boldsymbol{X}}) + \frac{3v^2}{8}E\left[\left(\overline{\boldsymbol{X}}'S^{-1}\overline{\boldsymbol{X}}\right)^2\right] - \cdots\right. \\ &\quad + \left.\frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)\cdots(-\frac{1}{2}-\overline{r-1})}{r!}v^r E\left[\left(\overline{\boldsymbol{X}}'S^{-1}\overline{\boldsymbol{X}}\right)^r\right] + \cdots\right\}. \end{aligned} \tag{31}$$

Now, since in our case p = 2,

$$\overline{\boldsymbol{X}}' S^{-1} \overline{\boldsymbol{X}} \sim \left(\frac{n-1}{n}\right) \times \frac{\chi_2^2(\lambda)}{\chi_{n-2}^2}, \text{ (Giri, 2004)},$$

$$\text{i.e. } \overline{\boldsymbol{X}}' S^{-1} \overline{\boldsymbol{X}} \sim \left(\frac{2(n-1)}{n(n-2)}\right) \times \frac{\frac{\chi_2^2(\lambda)}{2}}{\frac{\chi_{n-2}^2}{n-2}}$$

$$\text{i.e. } \overline{\boldsymbol{X}}' S^{-1} \overline{\boldsymbol{X}} \sim \left(\frac{2(n-1)}{n(n-2)}\right) \times F_{2,n-2}(\lambda),$$

where $\chi_2^2(\lambda)$ denotes the non central chi-square distribution with 2 degrees of freedom and non-centrality parameter $\lambda = n\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}$, χ_{n-2}^2 is the central chi-square distribution with $(n-2)$ degrees of freedom and $F_{2,n-2}(\lambda)$ denotes the noncentral F-distribution with "2" and (n-2) degrees of freedom and non-centrality parameter "λ". Note that the r^{th} raw moment of the non-central F-distribution exists only for $(n-p) > 2r$. In our case, p = 2. Also, in practice, to avoid some undesirable situations (e.g., over-fitting), the sample size (n) is considered to be large compared to the number of variables (i.e., number of quality characteristics), i.e., $n > p$ while, often very high values of "r" are not used in practice. Hence, it is logical to assume $(n-2) > 2r$ and this guarantees the existence of the raw moments of the non-central F-distribution.

Now, let $C_{n,2} = \frac{2(n-1)}{n(n-2)}$. Moreover, since observed value of $\overline{\boldsymbol{X}}' S^{-1} \overline{\boldsymbol{X}}$ is likely to be small, we take into account the raw moments of $\overline{\boldsymbol{X}}' S^{-1} \overline{\boldsymbol{X}}$ in (31) up to second order. Then, from (31),

$$E[\widehat{C}_{p,c}(0,v)] = k(n) \times C_{p,c}(0,0) \times \left\{1 - \frac{v}{4} \times C_{n,2} \times \frac{(n-2)(2+\lambda)}{n-4} + \frac{3v^2}{32} \times C_{n,2}^2 \times \left[\frac{(n-1)^2\left\{\lambda^2 + 8(\lambda+1)\right\}}{(n-4)\times(n-6)}\right]\right\} \tag{32}$$

and

$$E\left[\widehat{C}^2_{p,c}(0,v)\right] = \{k(n)\}^2 \times C^2_{p,c}(0,0) \times \left\{1 - \frac{v}{2} \times C_{n,2} \times \frac{(n-2)(2+\lambda)}{n-4} + \frac{v^2}{4} \times C_{n,2}^2 \times \left[\frac{(n-1)^2\left\{\lambda^2 + 8(\lambda+1)\right\}}{(n-4)\times(n-6)}\right]\right\}. \tag{33}$$

Hence, from (32) and (33), the variance of $\widehat{C}_{p,c}(0,v)$ will be

$$V[\widehat{C}_{p,c}(0,v)] = \{k(n)\}^2 \times C^2_{p,c}(0,0) \times \left\{1 - \frac{v}{2} \times C_{n,2} \times \frac{(n-2)(2+\lambda)}{n-4} + \frac{v^2}{4} \times C_{n,2}^2 \times \left[\frac{(n-1)^2\left\{\lambda^2 + 8(\lambda+1)\right\}}{(n-4)\times(n-6)}\right] - \left[1 - \frac{v}{4} \times C_{n,2} \times \frac{(n-2)(2+\lambda)}{n-4} + \frac{3v^2}{32} \times C_{n,2}^2\right.\right.$$

$$\times \frac{(n-1)^2\left\{\lambda^2+8(\lambda+1)\right\}}{(n-4)\times(n-6)}\Bigg]^2\Bigg\}. \tag{34}$$

Since $C_{p,c}(u, v)$ is a hybrid of $C_{p,c}(u, 0)$ and $C_{p,c}(0, v)$, for p = 2 and retaining up to second order raw moments of $\overline{\boldsymbol{X}}' S^{-1}\overline{\boldsymbol{X}}$, we have from (21)

$$\begin{aligned}
E[\widehat{C}_{p,c}(u,v)] &= \frac{1}{\sqrt{\chi^2_{\alpha,2}}} \times \left[E(|S|^{-\frac{1}{4}})\right] \times \left[\frac{D}{2} - \frac{u}{\sqrt{\pi}} \times E(\widehat{\boldsymbol{\mu}^*})\right] \times \left\{E\left[(1+v\overline{\boldsymbol{X}}'S^{-1}\overline{\boldsymbol{X}})^{-\frac{1}{2}}\right]\right\} \\
&= \frac{1}{\sqrt{\chi^2_{\alpha,2}}} \times \frac{\sqrt{n-1}}{|\Sigma|^{\frac{1}{4}}} \times \frac{\Gamma(\frac{2n-5}{2})}{\Gamma(n-2)} \times \left[\frac{D}{2} - \frac{u\sqrt{2}}{\pi} \times \Im(K)\sigma_1^*\right] \\
&\quad \times \left\{1 - \frac{v}{4} \times C_{n,2} \times \frac{(n-2)(2+\lambda)}{n-4} + \frac{3v^2}{32} \times C^2_{n,2}\right. \\
&\quad \left. \times \left[\frac{(n-1)^2\left\{\lambda^2+8(\lambda+1)\right\}}{(n-4)\times(n-6)}\right]\right\}
\end{aligned} \tag{35}$$

$$\begin{aligned}
E\left[\widehat{C}^2_{p,c}(u,v)\right] &= \frac{1}{\chi^2_{\alpha,2}} \times \frac{n-1}{(n-3)\sqrt{|\Sigma|}} \times \left\{\frac{D^2}{4} + \frac{u}{\pi}\sigma_1^* \times \left[u(2-k^2)\sigma_1^* - D\sqrt{2}\Im(k)\right]\right\} \\
&\quad \times \left\{1 - \frac{v}{2} \times C_{n,2} \times \frac{(n-2)(2+\lambda)}{n-4} + \frac{v^2}{4} \times C^2_{n,2}\right. \\
&\quad \left. \times \left[\frac{(n-1)^2\left[\lambda^2+8(\lambda+1)\right]}{(n-4)\times(n-6)}\right]\right\}
\end{aligned} \tag{36}$$

and hence, $V[\widehat{C}^2_{p,c}(u, v)]$ can be obtained from (35) and (36).

5. Comparative Study of the Performance of $C_{p,c}(u, v)$ and Other Existing Capability Indices for Circular Specification

To compare the performances of $C_{p,c}(u, v)$ for u = 0, 1 and v = 0, 1 with those of the existing ones, we use five data sets viz. data set used by Bothe (2006) and four simulated bi-variate normal data sets. Bothe's (2006) data (let us refer it as D^*) deals with an automobile supplier who bores a large hole near the middle of an aluminium flywheel housing. The "X_1" and "X_2" coordinates of the hole centers for 15 housings bores are measured and recorded (Bothe, 2006). Here the diameter of the circular specification region is 10 units, i.e., $D = 10$ and the target center is assumed to be at the (0, 0) point. From this data set, we have, $\widehat{\boldsymbol{\mu}} = \begin{pmatrix} 2.5 \\ 3.2 \end{pmatrix}$ and $\widehat{\Sigma} = \begin{pmatrix} 0.50 & 0.1428 \\ 0.1428 & 0.4571 \end{pmatrix}$ with $\widehat{\rho} = 0.2988$. We then simulate another four data sets (all with sample size n = 15) based on the summary statistics of D^* as follows.

1. Variance along the axis X_2 is increased 2.5 times (i.e., 250 %) and its new value is $\widehat{\sigma_2} = 1.1427$. All the other summary statistics remain unchanged. This will increase the difference between the variances along the two axis. We call this data set **D - I**.

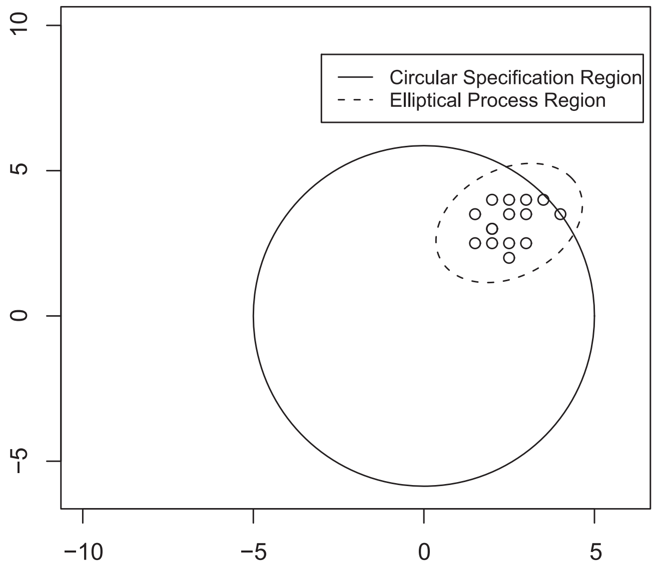

Figure 4. Specification region and process region corresponding to data set D*.

2. The correlation coefficient is increased to $\widehat{\rho} = 0.85$ from $\widehat{\rho} = 0.2988$. For this high value of $\widehat{\rho}$, the two axes will no longer be mutually perpendicular . All the other summary statistics remain unchanged. We call this data set **D - II**.
3. Variance along the axis X_2 is increased 2.5 times (i.e., 250%) and its new value is $\widehat{\sigma_2} = 1.1427$ and also the correlation coefficient is increased to be $\widehat{\rho} = 0.85$. All the other summary statistics remain unchanged. This will make the process region more elliptical than circular. We call this data set **D - III**.
4. Finally, we keep $\widehat{\Sigma}$ unchanged and consider the new $\widehat{\boldsymbol{\mu}}$ to be $(0.02, 0.02)'$. This will bring the process center closer to the target center viz. $(0, 0)'$. We call this data set **D - IV**.

The five sets of circular specification regions along with their corresponding process regions are given in Figs. 4–8.

The computed values of the various PCI's, threshold values of $\widehat{C}_{p,c}$ and the minimum observable as well as expected proportion of non-conformance are summarized in Table 1. Note that we have excluded two of the indices proposed by Bothe (2006) (see (5)) viz. $\widehat{C}_P$ and $\widehat{C}_{Pk}$ as those are specially designed to measure short term capability of a process and hence are not comparable to the rest of the PCI's.

To make the comparison proper we take the square root of $\widehat{PC}_p$ and $\widehat{PC}_{pk}$ values and denote them as $\widehat{PC}^*_p$ and $\widehat{PC}^*_{pk}$, respectively. From Table 1 and Figs. 4–8, one can observe the following.

1. The member indices of $\widehat{C}_{p,c}(u, v)$ namely, $\widehat{C}_{p,c}$, $\widehat{C}_{pk,c}$, $\widehat{C}_{pm,c}$, and $\widehat{C}_{pmk,c}$ and $\widehat{P}_p$ and $\widehat{P}_{pk}$ always decrease with the increase in the variances along either of the axes. Although, in the present set of examples, this is true for $\widehat{PC}_p$ and $\widehat{PC}_{pk}$ as well, this may not always be true for these two indices, e.g., when the smaller variance gets increased but still remains less than the other one ensuring that the maximum of the variances remain unchanged.
2. $\widehat{C}_{p,c}(u, v)$ values increase with the increase in "$\widehat{\rho}$".

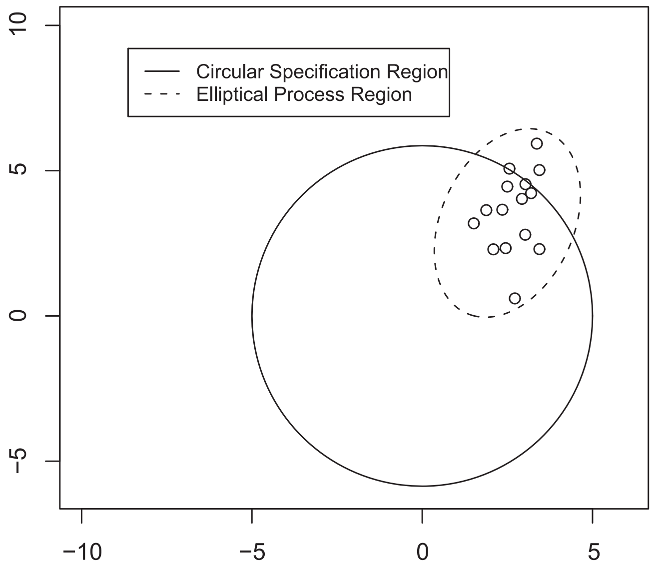

Figure 5. Specification region and process region corresponding to data set D - I.

Although Bothe (2006) assumed both the axes of the specification region to be independent, from (5) it can be seen that $\widehat{P}_p$ and $\widehat{P}_{pk}$ are based on $\widehat{\mu_c}$ and $\widehat{\mu_r}$, respectively, and hence on the entire data set. As such, $\widehat{P}_p$ and $\widehat{P}_{pk}$ are also affected by the non-zero correlation of the axes and that is why their values differ in cases of data sets D^* and D - II.

Since $\widehat{PC}_p$ and $\widehat{PC}_{pk}$ are not influenced by "$\widehat{\rho}$" or the actual observations as such, their values remain unchanged for the data sets D^*, D - II, and D - IV.

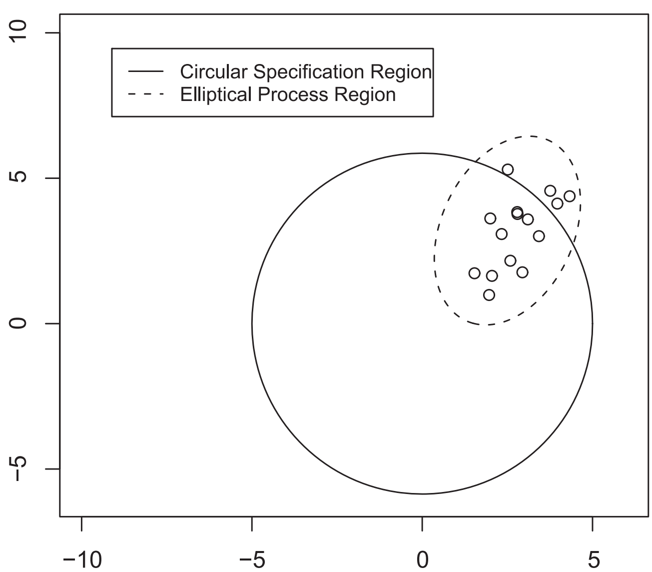

Figure 6. Specification region and process region corresponding to data set D - II.

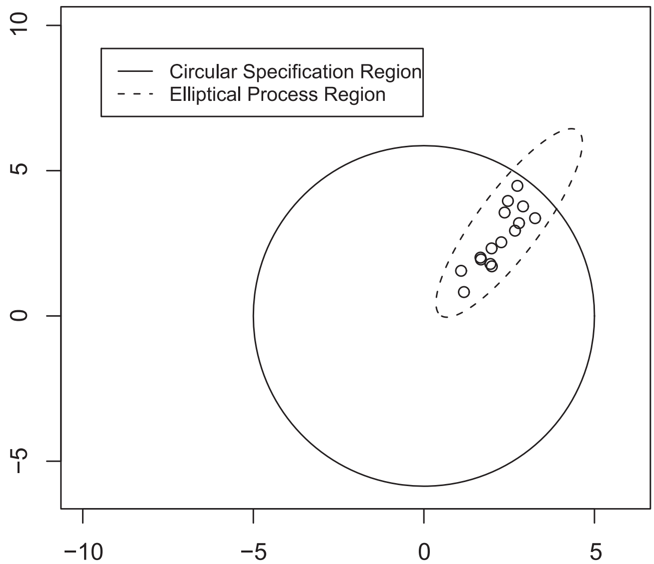

Figure 7. Specification region and process region corresponding to data set D - III.

3. The difference between the two data sets viz, D^* and D - IV is that, despite having the same dispersion structure, D^* is highly off-centered while the center of D - IV is in the close vicinity of the target center. Both $\widehat{C}_{p,c}$ and $\widehat{PC}_p$ remain unchanged for D^* and D - IV despite the fact that the process centering is changed in D - IV. Hence, $\widehat{C}_{p,c}$ and $\widehat{PC}_p$ indeed measure the potential capability of a process.

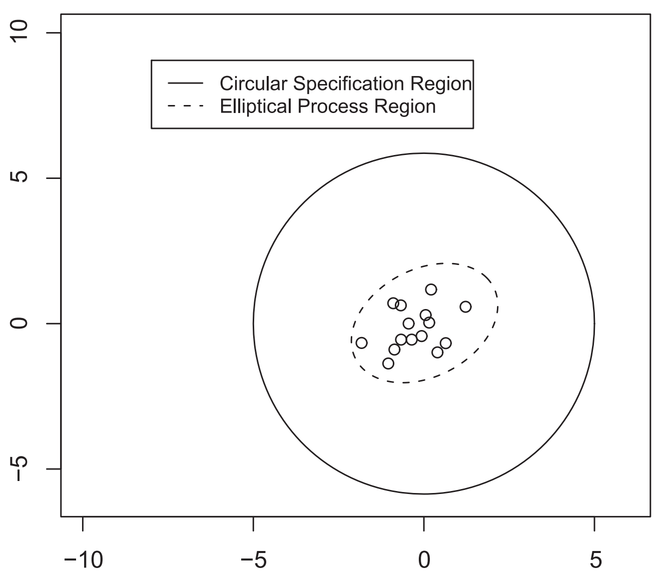

Figure 8. Specification region and process region corresponding to data set D - IV.

Table 1
Comparative study of the performances of various PCI's under circular specification region

	D^*	D - I	D - II	D - III	D - IV
$\widehat{C}_{p,c}$	2.4391	1.9398	3.2834	2.6109	2.4391
$\widehat{C}^T_{p,c}$	0.9173	0.8975	1.2346	1.2346	2.1736
$\widehat{C}_{pk,c}$	2.1943	1.7102	3.0015	2.2735	2.1736
$\widehat{C}_{pm,c}$	0.4582	0.4085	0.6666	0.4793	2.4376
$\widehat{C}_{pmk,c}$	0.4122	0.4073	0.6094	0.4173	2.1722
$\widehat{P}_{NC}$	8×10^{-7}	3.2×10^{-6}	1.2×10^{-12}	1.3×10^{-5}	8.5×10^{-7}
$\widehat{P}^E_{NC}$	0.119	0.0939	0.00478	0.10749	0.01044
$\widehat{P}_p$	4.03	1.9050	2.5153	1.7507	2.4848
$\widehat{P}_{pk}$	0.38	0.4014	0.3243	0.4155	2.5191
$\widehat{PC}_p$	5.5556	2.4309	5.5556	2.4309	5.5556
$\widehat{PC}^*_p$	2.357	1.5591	2.357	1.5591	2.357
$\widehat{PC}_{pk}$	0.6541	0.4733	0.6541	0.4733	5.4103
$\widehat{PC}^*_{pk}$	0.8088	0.6880	0.8088	0.6880	2.3260

However, although Bothe (2006) designed $\widehat{P}_p$ to measure potential capability of a process, since its definition includes $\widehat{\mu_c}$, $\widehat{P}_p$ fails to measure the potential capability and this is evident from the tabulated values of $\widehat{P}_p$ in Table 1 also, as its values for D^* and D - II are not the same.

4. Both $\widehat{C}_{p,c}$ and $\widehat{PC}_p$ conclude that all the processes described in Table 1 are potentially capable though $\widehat{PC}_p$ somewhat exaggerates the situation.
5. $\widehat{C}^T_{p,c}$ value remain unchanged for D^* and D - IV (as desired), since for those two data sets $\widehat{\sigma}_{\min} = \widehat{\sigma}_1$ and $\widehat{\rho}$ is fixed at 0.85.
6. $\widehat{C}_{pk,c}$ possesses a very interesting characteristic in a sense that similar to $\widehat{C}_{pk}$ of symmetric bilateral specification, it does not measure the proximity of the process center towards the target center (Boyles, 1991). Rather, it measures the expected proportion of non-conformance ($\widehat{P}^E_{NC}$) under the prevailing process setup. As can be seen in Table 1, the $\widehat{C}_{pk,c}$ values are always very high compared to the corresponding $\widehat{C}_{pm,c}$ values which may apparently look like conflicting outputs. However, the $\widehat{P}^E_{NC}$ values, expressed as a function of $\widehat{C}_{pk,c}$, wipes out this conflict generating similar decision regarding the performance of the process as $\widehat{C}_{pm,c}$ does and here lies the efficacy of $\widehat{C}_{pk,c}$.
7. $\widehat{C}_{pm,c}$ measures the proximity of the process center towards the target center efficiently and rightly concludes that all the processes except that described under D - IV are incapable as they suffer from considerably high off-centering.
8. $\widehat{C}_{pmk,c}$, which gives equal weightage to both the process nonconformance and the proximity between the process and the specification center, also considers only D - IV to be capable as other processes, apart from being off-centered, produce considerably high $\widehat{P}_{NC}$.
9. $\widehat{P}_{NC}$ and $\widehat{P}^E_{NC}$ are highly sensitive to change in process centering and process dispersion and this is also evident from the diagrammatic representations of the specification and process regions of the five data sets as given in Figs. 4–8.

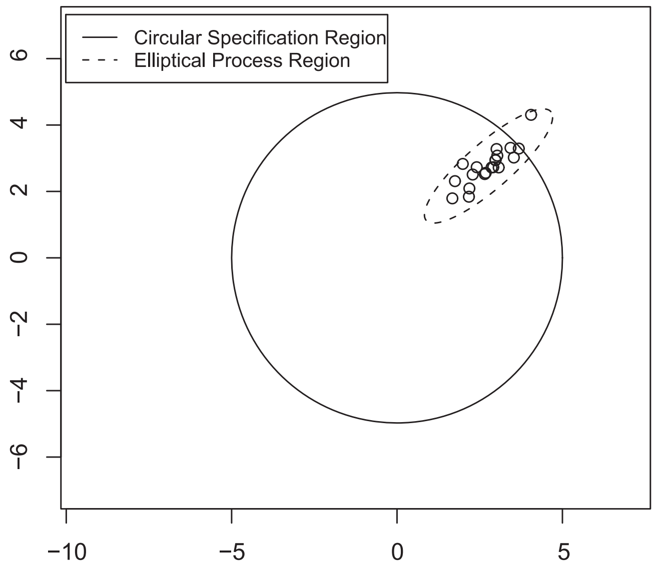

Figure 9. Plot for data on striker.

10. One interesting aspect of the data set D - III is revealed in Fig. 7. Although all the observations corresponding to this data set are within the circular specification region, the expected proportion of non-conformance is found to be 0.10749 (see Table 1), i.e., with the present process centering and the dispersion scenario, the process is likely to produce 10.749% items beyond either of the specification limits. This paradox is solved by Fig. 7 which shows that a considerable portion of the elliptical process region is outside the circular specification region. Hence, plotting merely the observations with respect to the specification region is not sufficient—one should draw the process region as well to have a better insight about the process.

6. A Practical Example

We are now in a position to apply our proposed superstructure of process capability indices for circular specification viz., $C_{p,c}(u, v)$ to real life data. The data used in the present context pertains to a product called striker used in making ammunitions. These parts should be properly assembled to ensure satisfactory performance of the produced ammunitions. Under such circumstances, holes drilled in strikers is our subject of interest.

The data on the coordinates (X_1, X_2) of the centers of each of the 20 randomly selected drilled holes are collected. The data is given in Table 2.

Based on past experience and various technical reasons, the diameter of the circular specification region has been set at 10 centimeter (cm.) i.e. here, $D = 10$. The summary statistics corresponding to the data in Table 2 are as follows:

$$\overline{X}_1 = 2.766, \overline{X}_2 = 2.776, \widehat{\sigma}_1^2 = 0.408, \widehat{\sigma}_2^2 = 0.321, \widehat{\rho} = 0.856.$$

Note that here X_1 and X_2 are highly correlated and hence $C_{p,c}(u, v)$ will be appropriate to apply. Figure 9 shows the plotted data, the corresponding elliptical process region and the circular specification region.

Table 2
Data on the co-ordinates of the centers of 20 drilled holes

Sl. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
X_1	2.65	3.01	2.86	1.99	1.75	2.68	3.68	2.91	4.05	2.17	2.98	3.03	1.67	3.53	2.29	3.43	2.19	2.97	2.41	3.07
X_2	2.52	3.28	2.72	2.83	2.31	2.56	3.29	2.73	4.31	1.84	2.94	3.08	1.79	3.02	2.51	3.31	2.09	2.94	2.73	2.72

The computed values of the member indices of the superstructure $C_{p,c}(u, v)$ as well as the existing PCI's for circular specification are as follows:

$\widehat{C}_{p,c} = 3.8097$, $\widehat{C}_{pk,c} = 3.5184$, $\widehat{C}_{pm,c} = 0.7605$, $\widehat{C}_{pmk,c} = 0.7024$, $\widehat{P}_p = 2.7992$, $\widehat{P}_{pk} = 0.4210$, $\widehat{PC}_p = 6.8152$, $\widehat{PC}^*_p = 2.6106$, $\widehat{PC}_{pk} = 0.7363$, $\widehat{PC}^*_{pk} = 0.8581$.

Note that, here $\widehat{C}_{p,c}$ value is considerably high as both the $\widehat{\sigma}_1^2$ and $\widehat{\sigma}_2^2$ are quite small. Also, $\widehat{C}^T_{p,c} = 1.3613$ which indicates that the process is "potentially capable". However, as can be seen from Fig. 9, the data points lie far away from the target center (0, 0) and this has been reflected by the low value of $\widehat{C}_{pm,c}$. But $\widehat{C}_{pk,c}$ fails to capture this characteristic of the data. Finally, $\widehat{C}_{pmk,c}$ being the combination of $\widehat{C}_{pk,c}$ and $\widehat{C}_{pm,c}$, also has low value but not lower than $\widehat{C}_{pm,c}$. Here, the minimum observable proportion of non-conformance is 2×10^{-8} but since the process is highly off-centered, at present, the expected proportion of non-conformance is 0.0598, i.e., about 6% of the drilled holes are not likely to meet the specification. Therefore, although the process is potentially capable, at present it is highly off-centered and hence its actual capability is not satisfactory.

In this context, although both $\widehat{P}_{pk}$ and $\widehat{PC}_{pk}$ rightly conclude the process to be incapable, since their basic assumptions (viz., homoscedasticity and mutual independence of the two axes) are not satisfied by the present data set, these indices are not suitable here. Moreover, despite being generalization of $\widehat{C}_{pk}$ in (1), neither of these two indices are directly related to the proportion of nonconformance. On the contrary, computation of $\widehat{C}_{p,c}(u, v)$ does not require the said assumptions, measures potential capability of a process and is directly related to the proportion of non-conformance. It also takes care of both the process centering as well as process variability, as has already been discussed in Sec. 5. Therefore, $C_{p,c}(u, v)$ is more suitable for processes with circular specification regions having heteroscedastic variances and nonzero correlation coefficient along the two axes.

7. Conclusions

Process capability indices for circular specification region are required to assess capability of processes like drilling a hole or ballistic processes where the efficiency of hitting a target is of prime interest. So far, only a few indices have been proposed in this field since the inception of the subject. Moreover, most these indices make some assumptions (viz. homoscedasticity and mutual independence of both the axes) which are often not practically viable. In this article we have defined a superstructure of PCI's denoted by $C_{p,c}(u, v)$ which do not require such assumptions and hence is more robust to deal with. We have also studied some of the very crucial inferential properties like the expectations and variances of member indices. The threshold value of $C_{p,c}$ and its relation to other member indices of $C_{p,c}(u, v)$ have been established. We have also formulated the expressions for minimum observable as well as expected proportion of non-conformance as functions of $C_{p,c}$ and $C_{pk,c}$, respectively. This strengthens the utility of the superstructure in practice. Moreover, the theory of $C_{p,c}(u, v)$ is developed based on the distributional assumption of bi-variate normality. From computational viewpoint, checking of this distributional assumption is quite easy and can be done using packages like **mvnormtest** and **mvShapiroTest** of the open source software "R". Generalized Shapiro-Wilk test (see Liang et al., 2009) can also be used for the same purpose. In this context, since $C_{p,c}(u, v)$ has been designed specially for bi-variate normal data, it is not applicable to the processes with non-normal statistical distributions in general. However, since we have followed a very general approach, the concept may be applied for other distributions as well with necessary modifications. Finally, this super-structure is easy to calculate and easier to interpret and hence should be more acceptable to practioners.

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