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PRIME RADICAL AND RADICAL IDEAL IN TERNARY SEMIRING

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Abstract: The main purpose of this paper is to obtain some important properties of prime radical of an ideal in a ternary semiring. Some special properties of prime radical and radical ideal are also obtained in the case when the ideals are k-ideals and h-ideals.

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1. Introduction and Preliminaries

A non-empty set S together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition is said to be a *ternary semiring* [1] if $(S, +)$ is a commutative semigroup and the ternary multiplication satisfies the following: (i) $(abc)de = a(bcd)e = ab(cde)$,

(ii) $(a + b)cd = acd + bcd$,

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(iii) $a(b+c)d = abd + acd$,

(iv) $ab(c+d) = abc + abd$ for all $a, b, c, d, e \in S$.

The set $\mathbb{Z}^-$ of all negative integers and the set $\mathbb{Z}_0^-$ of all non positive integers are two natural examples of ternary semiring with usual addition and ternary multiplication. $M_2(\mathbb{Z}_0^-)$, the set of all 2×2 matrices over $\mathbb{Z}_0^-$ is also a ternary semiring with canonically defined operations. If there exist an element $0 \in S$ such that $x + 0 = x$ and $0xy = x0y = xy0 = 0$ for all $x, y \in S$ then '0' is called the *zero element* or simply the *zero* [1] of the ternary semiring S and S is called a ternary semiring with zero. An element $e \in S$ is said to be a *unital element* [1] if $eex = exe = xee$ for all $x \in S$. The ternary semiring S is said to be *commutative* if $x_1x_2x_3 = x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$ for all $x_1, x_2, x_3 \in S$ and $\sigma \in S_3$. $\mathbb{Z}_0^-$ is a ternary semiring with zero, with a unital element and which is commutative as well. An additive subsemigroup T of S is called a *ternary subsemiring* [1] of S if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$. An additive subsemigroup I of S is called a *left* or *right* or a *lateral ideal* [4] of S if $s_1s_2i \in I$ or $is_1s_2 \in I$ or $s_1is_2 \in I$ respectively for all $s_1, s_2 \in S$ and for all $i \in I$. If I is a left, a right, a lateral ideal of S, then I is called an *ideal* [4] of S. In the ternary semiring $M_2(\mathbb{Z}_0^-)$ the set $I = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in 2\mathbb{Z}_0^- \right\}$ is an ideal. An ideal I of the ternary semiring S is said to be a *k-ideal* [1] if for $a, b \in S$, $a + b \in I$ and $a \in I$ then $b \in I$. An ideal I of the ternary semiring S is said to be an *h-ideal* [1] if for $x \in S$ and for $i_1, i_2 \in I$, $x + i_1 + s = i_2 + s$, $s \in S$ implies that $x \in I$. A proper ideal P of S is called a *prime ideal* [3] of S if for any three ideals A, B, C of S, $ABC \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$. In the commutative ternary semiring $\mathbb{Z}_0^-$ of all non positive integers, the ideal $I = \{3k : k \in \mathbb{Z}_0^-\}$ is a prime ideal. A proper ideal Q of S is called a *semiprime ideal* [5] of S if $I^3 \subseteq Q$ implies $I \subseteq Q$ for any ideal I of S. In the commutative ternary semiring $\mathbb{Z}_0^-$ of all non positive integers the ideal $Q = \{6k : k \in \mathbb{Z}_0^-\}$ is a semiprime ideal. It may be noted that every prime ideal in a ternary semiring is a semiprime ideal but converse is not true. For example, in the commutative ternary semiring $\mathbb{Z}_0^-$ of all negative integers with zero the semiprime ideal $Q = \{6k : k \in \mathbb{Z}_0^-\}$ is not a prime ideal. A nonempty subset M of S is called an *m-system* [3] if for each $a, b, c \in M$ there exist elements x_1, x_2, x_3, x_4 of S such that $ax_1bx_2c \in M$ or $ax_1x_2bx_3x_4c \in M$ or $ax_1x_2bx_3cx_4 \in M$ or $x_1ax_2bx_3x_4c \in M$.

The notion of ternary semiring was introduced by T. K. Dutta and S. Kar in [1] in the year 2003, as a natural generalization of ternary ring, introduced by W. G. Lister [6] in 1971. It is also found to be a generalization of semiring. The notion of prime radical [7] of an ideal is important in the theory of rings, as well as of semirings. In this paper we study prime radicals in ternary semiring as stated in the abstract. Among others we obtain an elementwise characterization of prime radical of an ideal (cf. Theorem 2.10) and a prime decomposition of radical ideals (cf. Theorem 2.21).

2. Prime Radical of an Ideal

Definition 2.1. [5] Let S be a ternary semiring and A be an ideal of S. Then *prime radical* of A, denoted by $\beta(A)$, is defined to be the intersection of all prime ideals of S each of which contains A.

Definition 2.2. An ideal L in a ternary semiring S is said to be a *nilpotent ideal* if $L^{2n+1} = 0$ for some integer $n \geq 0$.

Theorem 2.3. *[3] In a ternary semiring S the following conditions are equivalent:*

(i) P is a prime ideal of S

(ii) $aSbSc \subseteq P$, $aSSbSSc \subseteq P$, $aSSbScS \subseteq P$, $SaSbSSc \subseteq P$ implies $a \in P$ or $b \in P$ or $c \in P$

(iii) $\langle a\rangle \langle b\rangle \langle c\rangle \subseteq P$ implies $a \in P$ or $b \in P$ or $c \in P$.

Corollary 2.4. *[3] An ideal I of a commutative ternary semiring S is prime if and only if $abc \in P$ implies that $a \in P$ or $b \in P$ or $c \in P$ for all elements $a, b, c \in S$.*

Now we deduce below some properties of $\beta(A)$.

Proposition 2.5. *For an ideal A of a ternary semiring S we have the following:*

(i) $A \subseteq \beta(A)$

(ii) If P is a prime ideal of S then $A \subseteq P$ iff $\beta(A) \subseteq P$

(iii) If B is an ideal of S satisfying $A \subseteq B$ then $\beta(A) \subseteq \beta(B)$

(iv) $\beta(A)$ is a semiprime ideal of S

(v) $\beta(A) = \beta(A^{2n+1})$, n being an integer and $n \geq 0$

(vi) $\beta(A)$ contains every nilpotent ideal of S

(vii) $\beta(\beta(A)) = \beta(A)$.

Proof. (i), (ii) and (iii) follow immediately from the definition of prime radical.

(iv) Clearly $\beta(A)$ is an ideal of S. Let $C^3 \subseteq \beta(A)$, where C is an ideal of S. Now $\beta(A) = \bigcap\{P_i/A \subseteq P_i,\ P_i$ is a prime ideal in S$\}$. So $C^3 \subseteq P_i$, for all P_i. Then P_i, being prime, $C \subseteq P_i$, for all P_i. Therefore $C \subseteq \beta(A)$, proving $\beta(A)$ is a semiprime ideal of S.

(v) A being an ideal in S, $A^{2n+1} \subseteq A$, n being an integer and $n \geq 0$. Hence by (iii), $\beta(A^{2n+1}) \subseteq \beta(A)$. Let $x \in \beta(A)$. Now $\beta(A) = \bigcap\{P_i/P_i \supseteq A,\ P_i$ is a prime ideal in S$\}$. Then $x \in P_i$ for all P_i. If possible let $x \notin \beta(A^{2n+1})$. Then there exists a prime ideal Q in S such that $Q \supseteq A^{2n+1}$ and $x \notin Q$. Now Q being prime, $A^{2n+1} \subseteq Q$ implies that $A \subseteq Q$ whence Q is some P_i. This gives a contradiction. Therefore $x \in \beta(A^{2n+1})$. Consequently $\beta(A) = \beta(A^{2n+1})$.

(vi) Let L be a nilpotent ideal in S. Then $L^{2n+1} = \{0\}$, for some integer $n \geq 0$. Hence $L^{2n+1} \subseteq \beta(A)$. So $L^{2n+1} \subseteq P_i$ for all $P_i \supseteq A$ and P_i a prime ideal. Then $L \subseteq P_i$, for all P_i. Therefore $L \subseteq \beta(A)$.

(vii) By (i), $A \subseteq \beta(A)$. So by (iii), $\beta(A) \subseteq \beta(\beta(A))$. Let $x \in \beta(\beta(A))$ and $\{P_i\}_{i\in I}$ be the family of prime ideals in S such that $A \subseteq P_i$ for all $i \in I$. Then by definition $\beta(A) \subseteq P_i$ for all $i \in I$ whence $\beta(\beta(A)) \subseteq P_i$. Hence $x \in P_i$ for all $i \in I$ whence $x \in \beta(A)$. Therefore $\beta(\beta(A)) = \beta(A)$. □

We recall below the theorem characterizing the elements of $\beta(A)$ for its use in the sequel.

Theorem 2.6. *[5] Let A be an ideal in a ternary semiring S. Then $\beta(A) = \{s \in S/$ every m-system in S which contains s, has a nonempty intersection with $A\}$.*

Proposition 2.7. *Let A be an ideal in a ternary semiring S. If $a \in \beta(A)$ then there exists an integer $n \geq 0$ such that $a^{2n+1} \in A$.*

Proof. Let $a \in \beta(A)$. Then by Theorem 2.6, every m-system in S containing a has a nonempty intersection with A. We consider $M = \{a^{2n+1}$: n being an integer and $n \geq 0\}$. Then M is an m-system containing a. Therefore $M \cap A \neq \phi$. Then there exists an integer $n \geq 0$ such that $a^{2n+1} \in A$. □

Proposition 2.8. *Suppose S is a commutative ternary semiring and M is an m-system in S containing a. Then there exists an integer $n \geq 0$ such that $a^{2n+1}xy \in M$ where $x, y \in S$.*

Proof. We use below repeatedly the definition of m-system and commutativity of S.

Since $a \in M$, there exist x_1, x_2, x_3, x_4 in S such that $ax_1ax_2a \in M$ or $ax_1x_2ax_3x_4a \in M$ or $ax_1x_2ax_3ax_4 \in M$ or $x_1ax_2ax_3x_4a \in M$. It follows that $a(x_1ax_2)a \in M$ or S being commutative, $a^3x_1x_2 \in M$ or $a^3x_1x_2x_3x_4 \in M$.

Let $a^3x_1x_2 \in M$. Then there exist $x_5, x_6, x_7, x_8 \in S$ such that $a^5x_1x_2 x_5x_6 \in M$ or $a^5x_1x_2x_5x_6x_7x_8 \in M$.

Let $a^3x_1x_2x_3x_4 \in M$. Then there exist $y_1, y_2, y_3, y_4 \in S$ such that $a^5x_1x_2x_3 x_4y_1y_2 \in M$ or $a^5x_1x_2x_3x_4y_1y_2y_3y_4 \in M$.

Continuing in this way we get for each integer $n \geq 0$, $a^{2n+1}xy \in M$ for some $x, y \in S$. □

Proposition 2.9. *Let A be an ideal in a commutative ternary semiring S such that $a^{2n+1} \in A$, where $a \in S$, n is an integer and $n \geq 0$. Then $a \in \beta(A)$.*

Proof. Let M be any m-system in S containing a. Then by Proposition 2.8, $a^{2n+1}xy \in M$, for some $x, y \in S$.

As A is an ideal and $a^{2n+1} \in A$, $a^{2n+1}xy \in A$. So $M \cap A \neq \phi$.

Therefore by Theorem 2.6, $a \in \beta(A)$. □

Combination of Proposition 2.7 and 2.9 gives rise to the following result.

Theorem 2.10. *Suppose S is a commutative ternary semiring and A is an ideal of S. Then $\beta(A) = \{a \in S/a^{2n+1} \in A$ for some positive integer $n \geq 0\}$.*

Definition 2.11. An ideal A in a ternary semiring S is called a *prime radical ideal* if $\beta(A) = A$.

We simply say a *prime radical ideal* to be a *radical ideal.*

Proposition 2.12. *If A is an ideal in a ternary semiring S then the following are equivalent:*

(i) $\beta(A) = A$,

(ii) $a^{2n+1} \in A$ implies $a \in A$, where n is an integer and $n \geq 0$.

Proof. $(i) \Rightarrow (ii)$. Let $a^{2n+1} \in A$ then by Proposition 2.9, $a \in \beta(A) = A$, proving (ii).

$(ii) \Rightarrow (i)$. We know that $A \subseteq \beta(A)$. Let $t \in \beta(A)$. Then by Proposition 2.7 there exists an integer $n \geq 0$ such that $t^{2n+1} \in A$. Hence by hypothesis $t \in A$. Hence $\beta(A) \subseteq A$, proving $\beta(A) = A$. □

Definition 2.13. A k-ideal (h-ideal) in a ternary semiring S which also is a radical ideal is called a *radical k-ideal (h-ideal).*

Theorem 2.14. *Let I be a radical k-ideal of a commutative ternary semiring S and U, V be any two subsets of S. Then*

$$T = \{x \in S : xUV \subseteq I\}$$

is a radical k-ideal.

Proof. T is clearly an ideal of S. Now let $(x + y) \in T$ and $x \in T$, $y \in S$. Then $(x + y)uv \in I$ and $xuv \in I$ for all $u \in U$, and for all $v \in V$.

So $yuv \in I$ for all $u \in U$, and for all $v \in V$ as I is a k-ideal in S.

Hence $y \in T$. Consequently, T is a k-ideal in S.

Let $x^{2n+1} \in T$, for some integer $n \geq 0$. Then $x^{2n+1}uv \in I$ for all $u \in U, v \in V$, which implies $x^{2n+1}u^{2n+1}v^{2n+1} \in I$ for all $u \in U$ and for all $v \in V$, as I is an ideal of S.

Therefore $(xuv)^{2n+1} \in I$ for all $u \in U$ and for all $v \in V$.

So $xuv \in I$ for all $u \in U$ and for all $v \in V$, as I is a radical ideal. Thus $xUV \subseteq I$ and so $x \in T$.

Hence by Proposition 2.12, T is also a radical ideal and the theorem is proved. □

Theorem 2.15. *Let I be a radical h-ideal of a commutative ternary semiring S and U, V are any two subsets of S. Then*

$$T = \{x \in S : xUV \subseteq I\}$$

is a radical h-ideal.

Proof. Clearly T is an ideal of S. Now let $x \in S$ and $x + i_1 + s = i_2 + s$ for $s \in S$ and for $i_1, i_2 \in T$.

Then $(x + i_1 + s)uv = (i_2 + s)uv$ for all $u \in U$ and for all $v \in V$.

Therefore $xuv + i_1uv + suv = i_2uv + suv$ where $suv \in S$ and $i_1uv \in I$, $i_2uv \in I$. So $xuv \in I$ for all $u \in U$ and for all $v \in V$ as I is an h-ideal in S.

Hence $x \in T$. Consequently T is an h-ideal.

The proof of the part that T is a radical ideal is similar to that in Theorem 2.14 □

Theorem 2.16. *In a ternary semiring intersection of any collection of radical ideals is again a radical ideal.*

Proof. Let S be a ternary semiring and $\{T_i : i \in \Lambda\}$ be any collection of radical ideals in S. Then by definition 2.11, $\beta(T_i) = T_i$, for all $i \in \Lambda$. Now $\bigcap\limits_{i\in\Lambda} T_i \subseteq T_i$ for all $i \in \Lambda$. So by Proposition 2.5(iii), $\beta(\bigcap\limits_{i\in\Lambda} T_i) \subseteq T_i$, for all $i \in \Lambda$. Therefore $\beta(\bigcap\limits_{i\in\Lambda} T_i) \subseteq \bigcap\limits_{i\in\Lambda} T_i$. Again $\bigcap\limits_{i\in\Lambda} T_i \subseteq \beta(\bigcap\limits_{i\in\Lambda} T_i)$ (cf. Proposition 2.5(i)). Therefore $\beta(\bigcap\limits_{i\in\Lambda} T_i) = \bigcap\limits_{i\in\Lambda} T_i$, proving $\bigcap\limits_{i\in\Lambda} T_i$ is a radical ideal. □

Definition 2.17. Suppose S is a ternary semiring with a subsemiring A and for an ideal I, $P = I \cap A$ is an ideal. If there is another ideal J in S such that $J \supseteq I$ and $P = J \cap A$ then we say I can be enlarged to an ideal in S which also contracts to P.

Theorem 2.18. *[3] Let A be an m-system and N be an ideal of a ternary semiring S such that $N \cap A = \phi$. Then there exists a maximal ideal M of S containing N such that $M \cap A = \phi$. Moreover, M is a prime ideal of S.*

Theorem 2.19. *Let S be a commutative ternary semiring and A be a ternary subsemiring of S. Let I be a radical ideal of S such that $abc \in I$, $a \in A$, $b, c \in S$ imply either $a \in I$ or $b \in I$ or $c \in I$. Then $P = I \cap A$ is a prime ideal in A. Also I can be expressed as an intersection of prime ideals each of which contracts to P.*

Proof. Let $a, b, c \in A$ such that $abc \in P$. Then $abc \in I$. Therefore by hypothesis either $a \in I$ or $b \in I$ or $c \in I$. Hence either $a \in P$ or $b \in P$ or $c \in P$. So P becomes a prime ideal (cf. Corollary 2.4).

Let $X = \bigcap\{J/J$ is a prime ideal of S with $J \supseteq I$ and $J \cap A = P\}$. Then $I \subseteq X$. To prove the reverse inclusion, let $x \notin I$. Then the m-system $M = \{x\} \cup \{dx^{2n} : d \in A$ but $d \notin P$ and n is a positive integer $\}$ has empty intersection with I. Then by Theorem 2.18 there exists a maximal ideal $Q \supseteq I$ with $Q \cap M = \phi$, which is also prime.

Then $P \subseteq Q \cap A$. Again for $q \in Q \cap A$, $qx^2 \in Q$, Q being an ideal of S. It follows that $qx^2 \notin M$. This together with definition of M and that $q \in A$ implies that $q \in P$. Therefore $Q \cap A \subseteq P$. Hence $P = Q \cap A$. Again $x \notin Q$ as $x \in M$ and $M \cap Q = \phi$. Therefore $x \notin X$ and so $X \subseteq I$. Consequently $I = X$. □

Definition 2.20. Let S be a ternary semiring and A be any subset of S. The radical ideal generated by A is denoted by $\{A\}$ and is defined to be the intersection of all radical ideals of S each of which contains A.

Clearly $\{A\}$ is the smallest radical ideal containing A.

For simplicity we write $\{A \cup \{a\}\}$ as $\{A, a\}$.

Theorem 2.21. *In a commutative ternary semiring S satisfying ascending chain condition on radical ideals any radical ideal is expressible as the intersection of finite number of prime ideals.*

Proof. Let S be a commutative ternary semiring satisfying ascending chain condition on radical ideals. Let X be the set of all radical ideals which cannot be expressed as the intersection of finite number of prime ideals and $X \neq \phi$. As S satisfies ascending chain condition on radical ideals, X has a maximal element say I. Since $I \in X$ and it cannot be expressed as intersection of finite number of prime ideals, I is not prime.

Therefore there exist $a, b, c \in S$ such that $abc \in I$ but $a \notin I$, $b \notin I$, $c \notin I$.

Then each of the radical ideals $\{I, a\}$, $\{I, b\}$, $\{I, c\}$ properly contains I.

Hence each of them is expressible as intersection of finite number of prime ideals in S.

Now $\{I, a\}\{I, b\}\{I, c\} \subseteq \{I, abc\} \subseteq I$. So for any $d \in \{I, a\} \cap \{I, b\} \cap \{I, c\}$, $d^3 \in I$ whence $d \in I$ as I is a radical ideal. So $\{I, a\} \cap \{I, b\} \cap \{I, c\} \subseteq I$.

Clearly $I \subseteq \{I, a\} \cap \{I, b\} \cap \{I, c\}$. So $I = \{I, a\} \cap \{I, b\} \cap \{I, c\}$. Therefore I can be expressed as an intersection of finite number of prime ideals, a contradiction. So $X = \phi$.

This completes the proof. $\square$

3. Concluding Remark

In order to study derivations in ternary semirings, in our next paper many of the results of this paper have been extensively used.

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