



Received: 04 September 2015
Accepted: 08 October 2015
Published: 29 October 2015

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Reviewing editor:
Lishan Liu, Qufu Normal University,
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PURE MATHEMATICS | RESEARCH ARTICLE

Sum and product theorems depending on the (p, q) -th order and (p, q) -th type of entire functions

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Abstract: The main object of the present paper is to obtain new estimates involving the (p, q) -th order and the (p, q) -th type of entire functions under some suitable conditions. Some open questions, which emerge naturally from this investigation, are also indicated as a further scope of study for the interested future researchers in this branch of Complex Analysis.

Subjects: Advanced Mathematics; Mathematics & Statistics; Science

Keywords: entire functions; maximum modulus; (p, q) -th order and (p, q) -th type; value distribution theory; relative growth of entire functions; property (A)

2010 Mathematics Subject classifications: Primary 30D35; Secondary 30D30

1. Introduction

A single-valued function of one complex variable, which is analytic in the finite complex plane, is called an entire (integral) function. For example, $\exp(z)$, $\sin z$, $\cos z$, and so on, are all entire functions. In the value distribution theory, one studies how an entire function assumes some values and the influence of assuming certain values in some specific manner on a function. In 1926, Rolf Nevanlinna initiated the value distribution theory of entire functions. This value distribution theory is a prominent branch of Complex Analysis and is the prime concern of this paper. Perhaps the *Fundamental Theorem of Classical Algebra*, which may be stated as follows:

If $P(z)$ is a non-constant polynomial in z with real or complex coefficients, then the equation $P(z) = 0$ has at least one root

ABOUT THE AUTHOR

Ever since the early 1960s, the first-named author of this paper has been engaged in researches in many different areas of Pure and Applied Mathematics. Some of the key areas of his current research and publication activities include (e.g.) Real and Complex Analysis, Fractional Calculus and Its Applications, Integral Equations and Transforms, Higher Transcendental Functions and Their Applications, q -Series and q -Polynomials, Analytic Number Theory, Analytic and Geometric Inequalities, Probability and Statistics and Inventory Modelling and Optimization.

This paper, dealing essentially with the order of growth and the type of entire (or integral) functions, is a step in the ongoing investigations in the value distribution theory (initiated by Rolf Nevanlinna in 1926), which happens to be a prominent branch of Complex Analysis.

PUBLIC INTEREST STATEMENT

The theory of Entire Functions (which are known also as Integral Functions) is potentially useful in a wide variety of areas in Pure and Applied Mathematical, Physical and Statistical Sciences. Indeed, a single-valued function of one complex variable, which is analytic in the finite complex plane, is called an entire (or integral) function. For example, some of the commonly used entire functions include such elementary functions as $\exp(z)$, $\sin z$, $\cos z$, and so on. In the value distribution theory, one studies how an entire function assumes some values and the influence of assuming certain values in some specific manner on a function. This investigation is motivated essentially by the fact that the determination of the order of growth and the type of entire functions is rather important with a view to studying the basic properties of the value distribution theory.

is the most well-known value distribution theorem. The value distribution theory deals with the various aspects of the behaviour of entire functions, one of which is the study of comparative growth properties of entire functions. For any entire function $f(z)$ given by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad (1.1)$$

we define the *maximum modulus* $M_f(r)$ of $f(z)$ on $|z| = r$ as a function of r by

$$M_f(r) = \max_{|z|=r} \{|f(z)|\}. \quad (1.2)$$

In this connection, for all sufficiently large values of r , we recall the following well-known inequalities relating the maximum moduli of any two entire functions $f_i(z)$ and $f_j(z)$:

$$M_{f_i \pm f_j}(r) < M_{f_i}(r) + M_{f_j}(r), \quad (1.3)$$

$$M_{f_i \pm f_j}(r) \geq M_{f_i}(r) - M_{f_j}(r) \quad (1.4)$$

and

$$M_{f_i \cdot f_j}(r) \leq M_{f_i}(r) \cdot M_{f_j}(r). \quad (1.5)$$

On the other hand, if we consider z_r to be a point on the circle $|z| = r$, we find for all sufficiently large values of r that

$$M_{f_i \cdot f_j}(r) = \max_{|z|=r} \{|f_i(z) \cdot f_j(z)|\} = \max_{|z|=r} \{|f_i(z)| \cdot |f_j(z)|\}, \quad (1.6)$$

which implies that

$$M_{f_i \cdot f_j}(r) \geq |f_i(z)| \cdot |f_j(z)|. \quad (1.7)$$

In terms of the maximum modulus $M_f(r)$ of the function $f(z)$, the order ρ_f of the entire function $f(z)$, which is generally useful for computational purposes, is defined by

$$\rho_f := \limsup_{r \rightarrow \infty} \left\{ \frac{\log \log M_f(r)}{\log r} \right\} \quad (0 \leq \rho_f \leq \infty). \quad (1.8)$$

Moreover, with a view to determining (e.g.) the relative growth of two entire functions with the same positive order, the type σ_f of the entire function $f(z)$ of order ρ_f ($0 < \rho_f < \infty$) is defined by

$$\sigma_f := \limsup_{r \rightarrow \infty} \left\{ \frac{\log M_f(r)}{r^{\rho_f}} \right\} \quad (0 \leq \sigma_f \leq \infty). \quad (1.9)$$

The determination of the order of growth and the type of entire functions is rather important in order to study the basic properties of the value distribution theory. In this regard, several researchers made extensive investigations on this subject and presented the following useful results.

THEOREM 1 (see Holland, 1973) *Let $f(z)$ and $g(z)$ be any two entire functions of orders ρ_f and ρ_g respectively. Then*

$$\rho_{f+g} = \rho_g \quad \text{when } \rho_f < \rho_g$$

and

$$\rho_{f \cdot g} \leq \rho_g \quad \text{when } \rho_f \leq \rho_g.$$

THEOREM 2 (see Levin, 1996) Let $f(z)$ and $g(z)$ be any two entire functions with orders ρ_f and ρ_g respectively. Then

$$\rho_{f+g} \leq \max \{ \rho_f, \rho_g \},$$

$$\rho_{f \cdot g} \leq \max \{ \rho_f, \rho_g \},$$

$$\sigma_{f+g} \leq \max \{ \sigma_f, \sigma_g \}$$

and

$$\sigma_{f \cdot g} \leq \sigma_f + \sigma_g.$$

By appropriately extending the notion of addition and multiplication theorems as introduced by Holland (1973) and Levin (1996), our main object in this paper is to give the corresponding extensions of Theorem A and Theorem B. In our present investigation, we make use of index-pairs and the concept of the (p, q) -th order of entire functions for any two positive integers p and q with $p \geq q$, which are introduced in Section 2. For the standard definitions, notations and conventions used in the theory of entire functions, the reader may refer to (e.g. Boas, 1957; Valiron, 1949). Several closely-related recent works on the subject of our present investigation include (e.g. Choi, Datta, Biswas, & Sen, 2015; Datta, Biswas, & Biswas, 2013, 2015; Datta, Biswas, & Sen, 2015).

2. Definitions, notations, and preliminaries

Let $f(z)$ be an entire function defined in the complex z -plane $\mathbb{C}$. Also let $M_f(r)$ denote the maximum modulus of $f(z)$ for $|z| = r$ ($0 < r < \infty$) as defined by (1.2). In our investigation, we use the following definitions, notations, and conventions:

$$\log^{[0]} x = x \quad \text{and} \quad \log^{[k]} x = \log \left(\log^{[k-1]} x \right) \quad (k \in \mathbb{N} := \{1, 2, 3, \dots\})$$

and

$$\exp^{[0]}(x) = x \quad \text{and} \quad \exp^{[k]}(x) = \exp \left(\exp^{[k-1]}(x) \right) \quad (k \in \mathbb{N}).$$

Sato (1963) introduced a more general concept of the order and the type of an entire function than those given by (1.8) and (1.9).

Definition 1 (see Sato, 1963) Let $l \in \mathbb{N} \setminus \{1\}$. The generalized order $\rho_f^{[l]}$ of an entire function $f(z)$ is defined by

$$\rho_f^{[l]} = \limsup_{r \rightarrow \infty} \left\{ \frac{\log^{[l]} M_f(r)}{\log r} \right\} \quad (l \in \mathbb{N} \setminus \{1\}; 0 < \rho_f^{[l]} < \infty). \quad (2.1)$$

Definition 2 (see Sato, 1963) Let $l \in \mathbb{N} \setminus \{1\}$. The generalized type $\sigma_f^{[l]}$ of an entire function $f(z)$ of the generalized order $\rho_f^{[l]}$ is defined by

$$\sigma_f^{[l]} = \limsup_{r \rightarrow \infty} \left\{ \frac{\log^{[l-1]} M_f(r)}{r^{\rho_f^{[l]}}} \right\} \quad (l \in \mathbb{N} \setminus \{1\}; 0 < \sigma_f^{[l]} < \infty). \quad (2.2)$$

Remark 1 When $l = 2$, Definitions 1 and 2 coincide with the Equations (1.8) and (1.9), respectively.

More recently, a further generalized concept of the (p, q) -th order and the (p, q) -th type of an entire function $f(z)$ was introduced by Juneja et al. (see Juneja, Kapoor, & Bajpai, 1976, 1977) as follows.

Definition 3 (see Juneja et al., 1976) Let $p, q \in \mathbb{N}$ ($p \geq q$). The (p, q) -th order $\rho_f(p, q)$ of an entire function $f(z)$ is defined by

$$\rho_f(p, q) = \limsup_{r \rightarrow \infty} \left\{ \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \right\} \quad (0 < \rho_f(p, q) < \infty). \quad (2.3)$$

Definition 4 (see Juneja et al., 1977) Let $p, q \in \mathbb{N}$ ($p \geq q$). The (p, q) -th type $\sigma_f(p, q)$ of an entire function $f(z)$ of the (p, q) -th order $\rho_f(p, q)$ ($b \leq \rho_f(p, q) \leq \infty$) is defined by

$$\sigma_f(p, q) = \limsup_{r \rightarrow \infty} \left\{ \frac{\log^{[p-1]} M_f(r)}{(\log^{[q-1]} r)^{\rho_f(p, q)}} \right\} \quad (0 < \sigma_f(p, q) < \infty), \quad (2.4)$$

where the parameter b is given by

$$b = \begin{cases} 1 & (p = q) \\ 0 & (p < q). \end{cases} \quad (2.5)$$

Remark 2 By comparing Definitions 3 and 4 with Definitions 1 and 2, respectively, it is easily observed that

$$\rho_f(l, 1) = \rho_f^{[l]} \quad \text{and} \quad \sigma_f(l, 1) = \sigma_f^{[l]}. \quad (2.6)$$

See also Remark 1 above.

Next, in connection with the above developments, we also recall the following definition.

Definition 5 (see Juneja et al., 1976) An entire function $f(z)$ is said to have the index-pair (p, q) ($p \geq q \geq 1$) if

$$b < \rho_f(p, q) < \infty$$

and $\rho_f(p-1, q-1)$ is not a nonzero finite number, where b is given by (2.5). Moreover, if

$$0 < \rho_f(p, q) < \infty,$$

then

$$\rho_f(p-n, q) = \infty \quad (n < p), \quad \rho_f(p, q-n) = 0 \quad (n < q)$$

and

$$\rho_f(p+n, q+n) = 1 \quad (n \in \mathbb{N}).$$

The following proposition will be needed in our investigation.

Proposition 1 Let $f_i(z)$ and $f_j(z)$ be any two entire functions with the index-pairs (p_i, q_i) and (p_j, q_j) , respectively. Then the following conditions may occur:

- (i) $p_i \geq p_j$, $q_i = q_j$ and $\rho_{f_i}(p_i, q_i) > \rho_{f_j}(p_j, q_j)$;
- (ii) $p_i \geq p_j$, $q_i < q_j$ and $\rho_{f_i}(p_i, q_i) = \rho_{f_j}(p_j, q_j)$;
- (iii) $p_i > p_j$, $q_i = q_j$ and $\rho_{f_i}(p_i, q_i) = \rho_{f_j}(p_j, q_j)$;

$$(iv) \quad p_i \geq p_j, \quad q_i < q_j \text{ and } \rho_{f_i}(p_i, q_i) > \rho_{f_j}(p_j, q_j);$$

$$(v) \quad p_i = p_j, \quad q_i = q_j \text{ and } \rho_{f_i}(p_i, q_i) = \rho_{f_j}(p_j, q_j);$$

$$(vi) \quad p_i = p_j, \quad q_i > q_j \text{ and } \rho_{f_i}(p_i, q_i) > \rho_{f_j}(p_j, q_j);$$

$$(vii) \quad p_i > p_j, \quad q_i < q_j \text{ and } \rho_{f_i}(p_i, q_i) < \rho_{f_j}(p_j, q_j);$$

$$(viii) \quad p_i > p_j, \quad q_i = q_j \text{ and } \rho_{f_i}(p_i, q_i) < \rho_{f_j}(p_j, q_j);$$

$$(ix) \quad p_i < p_j, \quad q_i < q_j \text{ and } \rho_{f_i}(p_i, q_i) > \rho_{f_j}(p_j, q_j);$$

$$(x) \quad p_i > p_j, \quad q_i > q_j \text{ and } \rho_{f_i}(p_i, q_i) \geq \rho_{f_j}(p_j, q_j).$$

The following definition will also be useful in our investigation.

Definition 6 (see Bernal-González, 1988) A non-constant entire function $f(z)$ is said to have the Property (A) if, for any $\sigma > 1$ and for all sufficiently large values of r , the following inequality holds true:

$$\left[M_f(r) \right]^2 \leq M_f(r^\sigma).$$

Remark 3 For examples of entire functions with or without the Property (A), one may see the earlier work (Bernal-González, 1988).

3. A set of Lemmas

Here, in this section, we present three lemmas which will be needed in the sequel.

LEMMA 1 (see Bernal-González, 1988) Suppose that $f(z)$ is an entire function, $\alpha > 1$, $0 < \beta < \alpha$, $s > 1$ and $0 < \mu < \lambda$. Then

$$(a) \quad M_f(\alpha r) > \beta M_f(r)$$

and

$$(b) \quad \lim_{r \rightarrow \infty} \left\{ \frac{M_f(r^s)}{M_f(r)} \right\} = \infty = \lim_{r \rightarrow \infty} \left\{ \frac{M_f(r^\lambda)}{M_f(r^\mu)} \right\}.$$

LEMMA 2 (see Bernal-González, 1988) Let $f(z)$ be an entire function which satisfies the Property (A). Then, for any integer $n \in \mathbb{N}$ and for all sufficiently large values of r ,

$$\left[M_f(r) \right]^n \leq M_f(r^\delta) \quad (\delta < 1).$$

LEMMA 3 (see Levin, 1980, p. 21) Let the function $f(z)$ be holomorphic in the circle $|z| = 2eR$ ($R > 0$) with $f(0) = 1$. Also let η be an arbitrary positive number not exceeding $\frac{3e}{2}$. Then, inside the circle $|z| = R$, but outside of a family of excluded circles, the sum of whose radii is not greater than $4\eta R$,

$$\log |f(z)| > -T(\eta) \log M_f(2eR),$$

where

$$T(\eta) = 2 + \log \left(\frac{3e}{2\eta} \right).$$

4. Main results

In this section, we state and prove the main results of this paper.

THEOREM 3 Let $f_i(z)$ and $f_j(z)$ be any two entire functions with index-pairs (p_i, q_i) and (p_j, q_j) , respectively, where $p_i, p_j, q_i, q_j \in \mathbb{N}$ are constrained by

$$p_i \geq q_i \quad \text{and} \quad p_j \geq q_j.$$

Then

$$\rho_{(f_i \pm f_j)}(p, q) \leq \max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\}, \quad (4.1)$$

where

$$p = \max \{ p_i, p_j \} \quad \text{and} \quad q = \min \{ q_i, q_j \}.$$

Equality in (4.1) holds true when any one of the first four conditions of the Proposition in Section 2 are satisfied for $i \neq j$.

Proof For

$$\rho_{(f_i \pm f_j)}(p, q) = 0,$$

the result (4.1) is obvious, so we suppose that

$$\rho_{(f_i \pm f_j)}(p, q) > 0.$$

Clearly, we can also assume that $\rho_{f_k}(p_k, q_k)$ is finite for $k = i, j$.

Now, for any arbitrary $\varepsilon > 0$, from Definition 3 for the (p_k, q_k) -th order, we find for all sufficiently large values of r that

$$M_{f_k}(r) \leq \exp^{[p_k]} \left[\left(\rho_{f_k}(p_k, q_k) + \varepsilon \right) \log^{[q_k]} r \right] \quad (k = i, j) \quad (4.2)$$

that is,

$$M_{f_k}(r) \leq \exp^{[\max \{ p_1, p_2 \}]} \left[\left(\max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\} + \varepsilon \right) \log^{[\min \{ q_1, q_2 \}]} r \right] \quad (k = i, j), \quad (4.3)$$

so that

$$M_{f_k}(r) \leq \exp^{[p]} \left[\left(\max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\} + \varepsilon \right) \log^{[q]} r \right] \quad (k = i, j). \quad (4.4)$$

Therefore, in view of (4.4), we deduce from (1.3) for all sufficiently large values of r that

$$M_{f_i \pm f_j}(r) < 2 \exp^{[p]} \left[\left(\max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\} + \varepsilon \right) \log^{[q]} r \right]. \quad (4.5)$$

Thus, by applying Lemma 1(a), we find from (4.5) for all sufficiently large values of r that

$$\frac{1}{2} M_{f_i \pm f_j}(r) < \exp^{[p]} \left[\left(\max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\} + \varepsilon \right) \log^{[q]} r \right],$$

that is,

$$M_{f_i \pm f_j} \left(\frac{r}{3} \right) < \exp^{[p]} \left[\left(\max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\} + \varepsilon \right) \log^{[q]} r \right],$$

that is,

$$\frac{\log^{[p]} M_{f_i \pm f_j} \left(\frac{r}{3} \right)}{\log^{[q]} \left(\frac{r}{3} \right) + O(1)} < \max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\} + \varepsilon.$$

Therefore, we have

$$\begin{aligned} \rho_{f_i \pm f_j}(p, q) &= \limsup_{r \rightarrow \infty} \left\{ \frac{\log^{[p]} M_{f_i \pm f_j} \left(\frac{r}{3} \right)}{\log^{[q]} \left(\frac{r}{3} \right) + O(1)} \right\} \\ &\leq \max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\} + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\rho_{f_i \pm f_j}(p, q) \leq \max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\}. \quad (4.6)$$

We now let any one of first four conditions of the Proposition in Section 2 be satisfied for $i \neq j$ ($i, j = 1, 2$). Then, since $\varepsilon > 0$ is arbitrary, from Definition 3 for the (p_k, q_k) -th order, we find for a sequence of values of r tending to infinity that

$$M_{f_k}(r) \geq \exp^{[p_k]} \left[\left(\rho_{f_k}(p_k, q_k) - \varepsilon \right) \log^{[q_k]} r \right] \quad (k = i, j). \quad (4.7)$$

Therefore, in view of the first four conditions of the Proposition in Section 2, we obtain for a sequence of values of r tending to infinity that

$$M_{f_i}(r) \geq \exp^{[p]} \left[\left(\max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\} - \varepsilon \right) \log^{[q]} r \right]. \quad (4.8)$$

We next consider the following expression:

$$\frac{\exp^{[p_i]} \left[\left(\rho_{f_i}(p_i, q_i) + \varepsilon \right) \log^{[q_i]} r \right]}{\exp^{[p_j]} \left[\left(\rho_{f_j}(p_j, q_j) + \varepsilon \right) \log^{[q_j]} r \right]} \quad (i \neq j). \quad (4.9)$$

By virtue of the first four conditions of the Proposition of Section 2 and Lemma 1(b), we find from (4.9) that

$$\lim_{r \rightarrow \infty} \left\{ \frac{\exp^{[p_i]} \left[\left(\rho_{f_i}(p_i, q_i) + \varepsilon \right) \log^{[q_i]} r \right]}{\exp^{[p_j]} \left[\left(\rho_{f_j}(p_j, q_j) + \varepsilon \right) \log^{[q_j]} r \right]} \right\} = \infty \quad (i \neq j). \quad (4.10)$$

Now, clearly, (4.10) can also be written as follows:

$$\lim_{r \rightarrow \infty} \left\{ \frac{\exp^{[p]} \left[\left(\max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\} - \varepsilon \right) \log^{[q]} r \right]}{\exp^{[p_j]} \left[\left(\rho_{f_j}(p_j, q_j) + \varepsilon \right) \log^{[q_j]} r \right]} \right\} = \infty, \quad (4.11)$$

where

$$p \geq p_j, \quad q \leq q_j \text{ and } \max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\} \geq \rho_{f_j}(p_j, q_j),$$

but all of the equalities do not hold true *simultaneously*. So, from (4.11), we find for all sufficiently large values of r that

$$\begin{aligned} & \exp^{[p]} \left[\left(\max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\} - \varepsilon \right) \log^{[q]} r \right] \\ & > 2 \exp^{[p_j]} \left[\left(\rho_{f_j}(p_j, q_j) + \varepsilon \right) \log^{[q_j]} r \right]. \end{aligned} \quad (4.12)$$

Thus, from (4.2), (4.8) and (4.12), we deduce for a sequence of values of r tending to infinity that

$$M_{f_i}(r) > 2 \exp^{[p_j]} \left[\left(\rho_{f_j}(p_j, q_j) + \varepsilon \right) \log^{[q_j]} r \right],$$

that is,

$$M_{f_i}(r) < 2M_{f_j}(r) \quad (i \neq j; i, j = 1, 2). \quad (4.13)$$

Therefore, from (4.8) and (4.13), and in view of Lemma 1(a) and (1.4), it follows for a sequence of values of r tending to infinity that

$$M_{f_i \pm f_j}(r) \geq M_{f_i}(r) - M_{f_j}(r) \quad (i \neq j),$$

that is,

$$M_{f_i \pm f_j}(r) \geq M_{f_i}(r) - \frac{1}{2}M_{f_i}(r) \quad (i \neq j),$$

that is,

$$M_{f_i \pm f_j}(r) \geq \frac{1}{2}M_{f_i}(r) \quad (i \neq j),$$

that is,

$$M_{f_i \pm f_j}(r) \geq \frac{1}{2} \exp^{[p]} \left[\left(\max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\} - \varepsilon \right) \log^{[q]} r \right],$$

so that

$$M_{f_i \pm f_j}(3r) \geq \exp^{[p]} \left[\left(\max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\} - \varepsilon \right) \log^{[q]} r \right],$$

which, for a sequence of values of r tending to infinity, yields

$$\frac{\log^{[p]} M_{f_i \pm f_j}(3r)}{\log^{[q]}(3r) + O(1)} \geq \left(\max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\} + \varepsilon \right),$$

that is,

$$\limsup_{r \rightarrow \infty} \left\{ \frac{\log^{[p]} M_{f_i \pm f_j}(3r)}{\log^{[q]}(3r) + O(1)} \right\} \geq \max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\},$$

so that

$$\begin{aligned} \rho_{f_i \pm f_j}(p, q) &= \limsup_{r \rightarrow \infty} \left\{ \frac{\log^{[p]} M_{f_i \pm f_j}(r)}{\log^{[q]} r} \right\} \\ &\geq \max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\}. \end{aligned} \quad (4.14)$$

Clearly, therefore, the conclusion of the second part of Theorem 1 follows from (4.6) and (4.14). $\square$

Remark 4 That the inequality sign in Theorem 1 cannot be removed is evident from Example 1 below.

Example 1 Given any two natural numbers l and m , the functions

$$f(z) = \exp^{[l]}(z^m) \text{ and } g(z) = -\exp^{[l]}(z^m)$$

have their maximum moduli given by

$$M_f(r) = \exp^{[l]}(r^m) \text{ and } M_g(r) = \exp^{[l]}(r^m),$$

respectively. Therefore, the following expressions:

$$\frac{\log^{[k]} M_f(r)}{\log r} \text{ and } \frac{\log^{[k]} M_g(r)}{\log r}$$

are both constants for each $k \in \mathbb{N} \setminus \{1\}$. Thus, obviously, it follows that

$$\rho_f^{[l+1]} = \rho_g^{[l+1]} = m,$$

but

$$\rho_f^{[k]} = \rho_g^{[k]} = \begin{cases} \infty & (2 \leq k \leq l) \\ 0 & (k > l+1). \end{cases}$$

Consequently, we have

$$\rho_{f+g}^{[l+1]} = 0 < \rho_f^{[l+1]} + \rho_g^{[l+1]} = 2m.$$

THEOREM 4 Let $f_i(z)$ and $f_j(z)$ be any two entire functions with index-pairs (p_i, q_i) and (p_j, q_j) , respectively, where $p_i, p_j, q_i, q_j \in \mathbb{N}$ are constrained by

$$p_i \geq q_i \text{ and } p_j \geq q_j.$$

Suppose also that $\rho_{f_i}(p_i, q_i)$ and $\rho_{f_j}(p_j, q_j)$ are both non-zero and finite. Then, for

$$p = \max \{p_i, p_j\} \text{ and } q = \min \{q_i, q_j\},$$

$$\sigma_{f_i \pm f_j}(p, q) = \sigma_{f_i}(p_i, q_i), \quad (4.15)$$

provided that any one of the first four conditions of the Proposition of Section 2 is satisfied for $i \neq j$.

Proof First of all, suppose that any one of the first four conditions of the Proposition of Section 2 is satisfied for $i \neq j$. Also let $\varepsilon > 0$ and $\varepsilon_1 > 0$ be chosen arbitrarily. Then, from Definition 4 for the (p_k, q_k) -type, we find for all sufficiently large values of r that

$$M_{f_k}(r) \geq \exp^{[p_k-1]} \left[(\sigma_{f_k}(p_k, q_k) + \varepsilon) (\log^{[q_k-1]} r)^{\rho_{f_k}(p_k, q_k)} \right] \quad (k = i, j). \quad (4.16)$$

Moreover, for a sequence of values of r tending to infinity, we obtain

$$M_{f_k}(r) \geq \exp^{[p_k-1]} \left[(\sigma_{f_k}(p_k, q_k) - \varepsilon) (\log^{[q_k-1]} r)^{\rho_{f_k}(p_k, q_k)} \right] \quad (k = i, j). \quad (4.17)$$

Therefore, from (1.3) and (4.16), we get for all sufficiently large values of r that

$$M_{f_i \pm f_j}(r) \leq \exp^{[p_i-1]} \left[(\sigma_{f_i}(p_i, q_i) + \varepsilon) (\log^{[q_i-1]} r)^{\rho_{f_i}(p_i, q_i)} \right] \times \left(1 + \frac{\exp^{[p_j-1]} \left[(\sigma_{f_j}(p_j, q_j) + \varepsilon) (\log^{[q_j-1]} r)^{\rho_{f_j}(p_j, q_j)} \right]}{\exp^{[p_i-1]} \left[(\sigma_{f_i}(p_i, q_i) + \varepsilon) (\log^{[q_i-1]} r)^{\rho_{f_i}(p_i, q_i)} \right]} \right) \quad (i \neq j). \quad (4.18)$$

Now, in light of any one of the first four conditions of the Proposition in Section 2, for all sufficiently large values of r , we can make the factor:

$$\left(1 + \frac{\exp^{[p_j-1]} \left[(\sigma_{f_j}(p_j, q_j) + \varepsilon) (\log^{[q_j-1]} r)^{\rho_{f_j}(p_j, q_j)} \right]}{\exp^{[p_i-1]} \left[(\sigma_{f_i}(p_i, q_i) + \varepsilon) (\log^{[q_i-1]} r)^{\rho_{f_i}(p_i, q_i)} \right]} \right) \quad (i \neq j),$$

which occurs on the right-hand side of (4.18), as small as possible. Hence, for any $\alpha > 1 + \varepsilon_1$, it follows from Lemma 1 (a) and (4.18) that

$$M_{f_i \pm f_j}(r) \leq \exp^{[p_i-1]} \left[(\sigma_{f_i}(p_i, q_i) + \varepsilon) (\log^{[q_i-1]} r)^{\rho_{f_i}(p_i, q_i)} \right] (1 + \varepsilon_1),$$

that is,

$$\left(\frac{1}{1 + \varepsilon_1} \right) M_{f_i \pm f_j}(r) \leq \exp^{[p_i-1]} \left[(\sigma_{f_i}(p_i, q_i) + \varepsilon) (\log^{[q_i-1]} r)^{\rho_{f_i}(p_i, q_i)} \right],$$

so that

$$M_{f_i \pm f_j}(r) \leq \exp^{[p_i-1]} \left[\alpha (\sigma_{f_i}(p_i, q_i) + \varepsilon) (\log^{[q_i-1]} r)^{\rho_{f_i}(p_i, q_i)} \right] \quad (4.19)$$

for all sufficiently large values of r . Thus, by using (4.19), we find for all sufficiently large values of r that

$$M_{f_i \pm f_j}(r) \leq \exp^{[p-1]} \left[\alpha (\sigma_{f_i}(p_i, q_i) + \varepsilon) (\log^{[q-1]} r)^{\max \{ \rho_{f_1}(p_1, q_1), \rho_{f_2}(p_2, q_2) \}} \right]. \quad (4.20)$$

Therefore, in view of Theorem 1, it follows from (4.20) that, for all sufficiently large values of r ,

$$\log^{[p-1]} M_{f_i \pm f_j}(r) \leq \alpha (\sigma_{f_i}(p_i, q_i) + \varepsilon) (\log^{[q-1]} r)^{\max \{ \rho_{f_1}(p_1, q_1), \rho_{f_2}(p_2, q_2) \}},$$

that is,

$$\frac{\log^{[p-1]} M_{f_i \pm f_j}(r)}{[\log^{[q-1]} r]^{\rho_{(f_i \pm f_j)}(p, q)}} \leq \frac{\alpha(\sigma_{f_i}(p_i, q_i) + \varepsilon) (\log^{[q-1]} r)^{\max\{\rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j)\}}}{[\log^{[q-1]} r]^{\max\{\rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j)\}}}.$$
 (4.21)

Hence, upon letting $\alpha \rightarrow 1+$ in (4.21), we find for all sufficiently large values of r that

$$\limsup_{r \rightarrow \infty} \left\{ \frac{\log^{[p-1]} M_{f_i \pm f_j}(r)}{(\log r)^{\rho_{(f_i \pm f_j)}(p, q)}} \right\} \leq \sigma_{f_i}(p_i, q_i),$$

that is,

$$\sigma^{(p, q)}(f_i \pm f_j) \leq \sigma^{(p_i, q_i)}(f_i).$$
 (4.22)

Again, from (1.4), (4.16) and (4.17), we see for a sequence of values of r tending to infinity that

$$M_{f_i \pm f_j}(r) \geq \exp^{[p_i-1]} \left[(\sigma_{f_i}(p_i, q_i) - \varepsilon) (\log^{[q_i-1]} r)^{\rho_{f_i}(p_i, q_i)} \right] \times \left(1 - \frac{\exp^{[p_j-1]} \left[(\sigma_{f_j}(p_j, q_j) + \varepsilon) (\log^{[q_j-1]} r)^{\rho_{f_j}(p_j, q_j)} \right]}{\exp^{[p_i-1]} \left[(\sigma_{f_i}(p_i, q_i) - \varepsilon) (\log^{[q_i-1]} r)^{\rho_{f_i}(p_i, q_i)} \right]} \right) \quad (i \neq j).$$
 (4.23)

Now, by virtue of any one of the first four conditions of the Proposition in Section 2, for all sufficiently large values of r , we can make the factor:

$$\left(1 - \frac{\exp^{[p_j-1]} \left[(\sigma_{f_j}(p_j, q_j) + \varepsilon) (\log^{[q_j-1]} r)^{\rho_{f_j}(p_j, q_j)} \right]}{\exp^{[p_i-1]} \left[(\sigma_{f_i}(p_i, q_i) - \varepsilon) (\log^{[q_i-1]} r)^{\rho_{f_i}(p_i, q_i)} \right]} \right) \quad (i \neq j),$$

which occurs on the right-hand side of (4.23), as small as possible. Hence, for any β constrained by

$$\beta > \frac{1}{1 - \varepsilon_1},$$

it follows from Lemma 1(a) and (4.23) that, for a sequence of values of r tending to infinity,

$$M_{f_i \pm f_j}(r) \geq \exp^{[p_i-1]} \left[(\sigma_{f_i}(p_i, q_i) - \varepsilon) (\log^{[q_i-1]} r)^{\rho_{f_i}(p_i, q_i)} \right] (1 - \varepsilon_1).$$

that is,

$$\left(\frac{1}{1 - \varepsilon_1} \right) M_{f_i \pm f_j}(r) \geq \exp^{[p_i-1]} \left[(\sigma_{f_i}(p_i, q_i) - \varepsilon) (\log^{[q_i-1]} r)^{\rho_{f_i}(p_i, q_i)} \right],$$

so that

$$M_{f_i \pm f_j}(\beta r) \geq \exp^{[p_i-1]} \left[(\sigma_{f_i}(p_i, q_i) - \varepsilon) (\log^{[q_i-1]} r)^{\rho_{f_i}(p_i, q_i)} \right].$$
 (4.24)

Therefore, by using (4.24), it follows for a sequence of values of r tending to infinity that

$$M_{f_i \pm f_j}(\beta r) \geq \exp^{[p-1]} \left[(\sigma_{f_i}(p_i, q_i) - \varepsilon) (\log^{[q-1]} r)^{\max\{\rho_{f_1}(p_1, q_1), \rho_{f_2}(p_2, q_2)\}} \right],$$

which, in the limit when $\beta \rightarrow 1+$, yields

$$\limsup_{r \rightarrow \infty} \left\{ \frac{\log^{[p-1]} M_{f_i \pm f_j}(r)}{(\log^{[q-1]} r)^{\max\{\rho_{f_1}(p_1, q_1), \rho_{f_2}(p_2, q_2)\}}} \right\} \geq \sigma_{f_i}(p_i, q_i). \quad (4.25)$$

Thus, in view of Theorem 1, we find from (4.25) that

$$\limsup_{r \rightarrow \infty} \left\{ \frac{\log^{[p-1]} M_{f_i \pm f_j}(r)}{(\log^{[q-1]} r)^{\rho_{(f_i \pm f_j)}(p, q)}} \right\} \geq \sigma_{f_i}(p_i, q_i),$$

that is,

$$\sigma^{(p, q)}(f_i \pm f_j) \geq \sigma^{(p_i, q_i)}(f_i). \quad (4.26)$$

Theorem 2 now follows from (4.22) and (4.26). $\square$

Our next result (Theorem 3) provides the condition under which the equality sign in the assertion (4.1) of Theorem 1 holds true in the case of the condition (v) of the Proposition of Section 2.

THEOREM 5 Let $f_1(z)$ and $f_2(z)$ be any two entire functions such that

$$\rho_{f_1}(p, q) = \rho_{f_2}(p, q) \quad (0 < \rho_{f_1}(p, q) = \rho_{f_2}(p, q) < \infty)$$

and

$$\sigma_{f_1}(p, q) \neq \sigma_{f_2}(p, q).$$

Then

$$\rho_{f_1 \pm f_2}(p, q) = \rho_{f_1}(p, q) = \rho_{f_2}(p, q) \quad (p, q \in \mathbb{N}; p \geq q). \quad (4.27)$$

Proof Under the hypotheses of Theorem 3, if we apply Theorem 1, it is easily seen that

$$\rho_{f_1 \pm f_2}(p, q) \leq \rho_{f_1}(p, q) = \rho_{f_2}(p, q).$$

Let us consider the case when

$$\rho_{f_1 \pm f_2}(p, q) < \rho_{f_1}(p, q) = \rho_{f_2}(p, q).$$

Then, in view of Theorem 2, we find that

$$\sigma_{f_1}(p, q) = \sigma_{f_1 \pm f_2 \mp f_2}(p, q) = \sigma_{f_2}(p, q),$$

which is a contradiction. Consequently, the assertion (4.27) of Theorem 3 holds true. $\square$

THEOREM 6 Let $f_i(z)$ and $f_j(z)$ be any two entire functions with the index-pairs (p_i, q_i) and (p_j, q_j) , respectively, for $p_i, p_j, q_i, q_j \in \mathbb{N}$ such that

$$p_i \geq q_i \quad \text{and} \quad p_j \geq q_j.$$

Then

$$\rho_{f_i, f_j}(p, q) \leq \max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\}, \quad (4.28)$$

where

$$p = \max \{p_i, p_j\} \quad \text{and} \quad q = \min \{q_i, q_j\}.$$

Equality in (4.28) holds true when any one of the first four conditions of the Proposition of Section 2 is satisfied for $i \neq j$. Furthermore, a similar relation holds true for the quotient

$$\tilde{f}(z) = \frac{f_j(z)}{f_i(z)},$$

provided that the function $\tilde{f}(z)$ is entire.

Proof Since the result is obvious when

$$\rho_{(f_i, f_j)}(p, q) = 0,$$

we suppose that $\rho_{(f_i, f_j)}(p, q) > 0$. Suppose also that

$$\max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\} = \rho.$$

We can clearly assume that $\rho_{f_k}(p_k, q_k)$ is finite for $k = i, j$.

Now, for any arbitrary $\varepsilon > 0$, we find from () that, for all sufficiently large values of r ,

$$M_{f_k}(r) \leq \exp^{[p]} \left[\left(\rho + \frac{\varepsilon}{2} \right) \log^{[q]} r \right] \quad (k = i, j). \quad (4.29)$$

We further consider the expression:

$$\frac{\exp^{[p-1]} \left[(\rho + \varepsilon) \log^{[q]} r \right]}{\exp^{[p-1]} \left[\left(\rho + \frac{\varepsilon}{2} \right) \log^{[q]} r \right]}$$

for all sufficiently large values of r . Thus, for any $\delta > 1$, it follows from the above expression that, for all sufficiently large values of $r \geq r_1 \geq r_0$,

$$\frac{\exp^{[p-1]} \left[(\rho + \varepsilon) \log^{[q]} r_0 \right]}{\exp^{[p-1]} \left[\left(\rho + \frac{\varepsilon}{2} \right) \log^{[q]} r_0 \right]} = \delta. \quad (4.30)$$

Next, in view of (4.29) and (1.5), we have

$$M_{f_i, f_j}(r) < \left[\exp^{[p]} \left[\left(\rho + \frac{\varepsilon}{2} \right) \log^{[q]} r \right] \right]^2 \quad (4.31)$$

for all sufficiently large values of r . Also, by applying Lemma 2, we find from (4.30) and (4.31) that, for all sufficiently large values of r ,

$$M_{f_i, f_j}(r) < \exp^{[p]} \left[\left(\rho + \frac{\varepsilon}{2} \right) \log^{[q]} r \right]^\delta,$$

that is,

$$M_{f_i, f_j}(r) < \exp^{[p]} \left[(\rho + \varepsilon) \log^{[q]} r \right].$$

Therefore, we have

$$\frac{\log^{[p]} M_{f_i f_j}(r)}{\log^{[q]} r} \leq (\rho + \varepsilon),$$

so that

$$\rho_{f_i f_j}(p, q) = \limsup_{r \rightarrow \infty} \frac{\log^{[p]} M_{f_i f_j}(r)}{\log^{[q]} r} \leq (\rho + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, it is easily observed that

$$\rho_{f_i f_j}(p, q) \leq \rho = \max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\}. \quad (4.32)$$

We now let any one of the first four conditions of the Proposition of Section 2 be satisfied for $i \neq j$. Then, without any loss of generality, we may assume that

$$f_k(0) = 1 \quad (k = i, j).$$

We may also suppose that $r > R$. Thus, from (4.7) and in view of the first four conditions of the Proposition of Section 2, we find for a sequence of values of R tending to infinity that

$$M_{f_i}(R) \geq \exp^{[p]} \left[(\rho - \varepsilon) \log^{[q]} R \right]. \quad (4.33)$$

Also, by using (4.4), we get for all sufficiently large values of r that

$$M_{f_j}(r) \leq \exp^{[p]} \left[(\rho + \varepsilon) \log^{[q]} r \right]. \quad (4.34)$$

In view of Lemma 3, if we take $f_j(z)$ for $f(z)$, $\eta = \frac{1}{16}$ and $2R$ for R , it follows that

$$\log |f_j(z)| > -T(\eta) \log M_{f_j}(2e \cdot 2R),$$

where

$$T(\eta) = 2 + \log \left(\frac{3e}{2 \cdot \frac{1}{16}} \right) = 2 + \log (24e).$$

Therefore, the following inequality:

$$\log |f_j(z)| > -(2 + \log (24e)) \log M_{f_j}(4e \cdot R)$$

holds true within and on the circle $|z| = 2R$, but outside of a family of excluded circles, the sum of whose radii is not greater than

$$4 \cdot \frac{1}{16} \cdot 2R = \frac{R}{2}.$$

If $r \in (R, 2R)$, then, on the circle $|z| = r$, we have

$$\log |f_j(z)| > -7 \log M_{f_j}(4e \cdot R). \quad (4.35)$$

Since $r > R$, we see from (4.33) that, for a sequence of values of r tending to infinity,

$$\begin{aligned} M_{f_i}(r) &> M_{f_i}(R) > \exp^{[p]} \left[(\rho - \varepsilon) \log^{[q]} R \right] \\ &> \exp^{[p]} \left[(\rho - \varepsilon) \log^{[q]} \left(\frac{r}{2} \right) \right]. \end{aligned} \quad (4.36)$$

We now let z_r be a point on the circle $|z| = r$ such that

$$M_{f_i}(r) = |f_i(z_r)|.$$

Then, since $r > R$, it follows from (1.5), (4.34), (4.35) and (4.36) that, for a sequence of values of r tending to infinity,

$$\begin{aligned} M_{f_i f_j}(r) &\geq |f_j(z_r)| M_{f_i}(r), \\ \text{that is,} \\ M_{f_i f_j}(r) &\geq \left[M_{f_j}(4eR) \right]^{-7} M_{f_i}(r), \end{aligned} \quad (4.37)$$

that is,

$$\begin{aligned} M_{f_i f_j}(r) &\geq \left(\exp^{[p]} \left[(\rho + \varepsilon) \log^{[q]} (4eR) \right] \right)^{-7} \cdot \exp^{[p]} \left[(\rho - \varepsilon) \log^{[q]} \left(\frac{r}{2} \right) \right], \\ \text{that is,} \\ M_{f_i f_j}(r) &\geq \left(\exp^{[p]} \left[(\rho + \varepsilon) \log^{[q]} (4er) \right] \right)^{-7} \cdot \exp^{[p]} \left[(\rho - \varepsilon) \log^{[q]} \left(\frac{4er}{8e} \right) \right]. \end{aligned} \quad (4.38)$$

Since

$$\lim_{r \rightarrow \infty} \left\{ \frac{\exp^{[p-1]} \left[(\rho - \varepsilon) \log^{[q]} \left(\frac{4er}{8e} \right) \right]}{\exp^{[p-1]} \left[(\rho + \varepsilon) \log^{[q]} (4er) \right]} \right\} = \infty,$$

we may observe, for all sufficiently large values of r with $r_n > r_1 > r_0$, that

$$\frac{\log \left[(\rho - \varepsilon) \log^{[q]} \left(\frac{4er_n}{8e} \right) \right]}{\log \left[(\rho + \varepsilon) \log^{[q]} (4er_n) \right]} < \frac{\log \left[(\rho - \varepsilon) \log^{[q]} \left(\frac{4er_0}{8e} \right) \right]}{\log \left[(\rho + \varepsilon) \log^{[q]} (4er_0) \right]} =: \delta.$$

Therefore, clearly, we have

$$\delta > 1.$$

Hence, for the above value of δ , we can easily verify that

$$\exp^{[p]} \left[(\rho - \varepsilon) \log^{[q]} \left(\frac{4er}{8e} \right) \right] \geq \exp^{[p]} \left[(\rho + \varepsilon) \log^{[q]} (4er) \right]^\delta. \quad (4.39)$$

Also, in light of Lemma 2, we find for all sufficiently large values of r that

$$\exp^{[p]} \left[(\rho + \varepsilon) \log^{[q]} (4er) \right]^\delta \geq \left(\exp^{[p]} \left[(\rho + \varepsilon) \log^{[q]} (4er) \right] \right)^8. \quad (4.40)$$

Now, from (4.38), (4.39) and (4.40), it follows for a sequence of values of r tending to infinity that

$$M_{f_i f_j}(r) \geq \exp^{[p]} \left[(\rho + \varepsilon) \log^{[q]} (4er) \right],$$

that is,

$$\frac{\log^{[p]} M_{f_i, f_j}(r)}{\log^{[q]} r + O(1)} \geq \rho + \varepsilon,$$

so that

$$\begin{aligned} \rho_{f_i, f_j}(p, q) &= \limsup_{r \rightarrow \infty} \left\{ \frac{\log^{[p]} M_{f_i, f_j}(r)}{\log^{[q]} r} \right\} \\ &\leq \rho = \max \left\{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \right\}. \end{aligned} \quad (4.41)$$

Consequently, the second part of Theorem 4 follows from (4.32) and (4.41).

We may next suppose that

$$f_k(z) = \frac{f_j(z)}{f_i(z)} \quad (i \neq j).$$

We also assume that any one of the conditions as laid down in the Proposition of Section 2 are satisfied for $i \neq j$. Therefore, we can write

$$f_j(z) = f_k(z) \cdot f_i(z).$$

If possible, let any one of the first four conditions of the Proposition of Section 2 is satisfied after replacing all i by k and all j by i in the first four conditions of the Proposition. We then find that

$$\rho_{f_j}(p_j, q_j) = \rho_{f_k}(p_k, q_k).$$

Consequently, the first four conditions of the Proposition reduce to the following forms for $i \neq j$:

- (i) $p_i \leq p_j$, $q_i = q_j$ and $\rho_{f_i}(p_i, q_i) < \rho_{f_j}(p_j, q_j)$;
- (ii) $p_i \leq p_j$, $q_i > q_j$ and $\rho_{f_i}(p_i, q_i) = \rho_{f_j}(p_j, q_j)$;
- (iii) $p_i < p_j$, $q_i = q_j$ and $\rho_{f_i}(p_i, q_i) = \rho_{f_j}(p_j, q_j)$;
- (iv) $p_i \leq p_j$, $q_i > q_j$ and $\rho_{f_i}(p_i, q_i) < \rho_{f_j}(p_j, q_j)$.

This evidently contradicts the hypothesis that any one of the conditions as laid down in the Proposition is satisfied for $i \neq j$. Therefore, our assumption about the possibility that any one of the first four conditions of the Proposition is satisfied after replacing all i by k and all j by i in the first four conditions of the Proposition is not valid. Thus, accordingly, any one of the above four conditions is satisfied if we replace all i by k and all j by i . Therefore, we have

$$\rho_{f_k}(p_k, q_k) = \rho_{\frac{f_j}{f_i}}(p_k, q_k) \leq \rho_{f_i}(p_i, q_i) = \rho.$$

Further, if possible, let any one of the first four conditions of the Proposition is satisfied after replacing all j by k only in the first four conditions of the Proposition.

Then

$$\rho_{f_j}(p_j, q_j) = \rho = \rho_{f_i}(p_i, q_i).$$

Thus, accordingly, the first four conditions of the Proposition reduces to the following forms for $i \neq j$:

- (i) $p_i \leq p_j$, $q_i = q_j$ and $\rho_{f_i}(p_i, q_i) < \rho_{f_j}(p_j, q_j)$;
- (ii) $p_i \leq p_j$, $q_i > q_j$ and $\rho_{f_i}(p_i, q_i) = \rho_{f_j}(p_j, q_j)$;
- (iii) $p_i < p_j$, $q_i = q_j$ and $\rho_{f_i}(p_i, q_i) = \rho_{f_j}(p_j, q_j)$;
- (iv) $p_i \leq p_j$, $q_i > q_j$ and $\rho_{f_i}(p_i, q_i) < \rho_{f_j}(p_j, q_j)$.

This also leads to a contradiction. Therefore, any one of the above four conditions is satisfied only after replacing all j by k . We thus obtain

$$\rho_{f_k}(p_k, q_k) = \rho_{f_i}(p_k, q_k) = \rho.$$

Our demonstration of Theorem 4 is evidently completed. □

Remark 5 Example 2 shows that the inequality sign in the assertion (4.28) of Theorem 4 cannot be removed.

Example 2 For $k, n \in \mathbb{N}$, the functions

$$f(z) = \exp^{[k]}(z^n) \text{ and } g(z) = \exp^{[k]}(-z^n)$$

have their maximum moduli given by

$$M_f(r) = \exp^{[k]}(r^n) \text{ and } M_g(r) = \exp^{[k]}(-r^n),$$

respectively. Therefore, we have

$$\frac{\log^{[l]} M_f(r)}{\log r} \text{ and } \frac{\log^{[l]} M_g(r)}{\log r}$$

are both constants for each $l \in \mathbb{N} \setminus \{1\}$. Thus, it follows that

$$\rho_f^{[k+1]} = \rho_g^{[k+1]} = n,$$

but

$$\rho_f^{[l]} = \rho_g^{[l]} = \infty \quad (2 \leq l \leq k)$$

and

$$\rho_f^{[l]} = \rho_g^{[l]} = 0 \quad (l > k+1).$$

Hence, we have

$$\rho_{f \cdot g}^{[k+1]} = 0 < \rho_f^{[k+1]} + \rho_g^{[k+1]} = 2n.$$

THEOREM 7 Let $f_i(z)$ and $f_j(z)$ be any two entire functions with the index-pairs (p_i, q_i) and (p_j, q_j) , respectively, for $p_i, p_j, q_i, q_j \in \mathbb{N}$ such that

$$p_i \geq q_i \quad \text{and} \quad p_j \geq q_j.$$

Suppose also that

$$\rho_{f_i}(p_i, q_i) \quad \text{and} \quad \rho_{f_j}(p_j, q_j)$$

are both non-zero and finite. Then, for

$$p = \max \{p_i, p_j\} \quad \text{and} \quad q = \min \{q_i, q_j\},$$

$$\sigma_{f_i, f_j}(p, q) = \sigma_{f_i}(p_i, q_i),$$

provided that any one of the first four conditions of the Proposition of Section 2 is satisfied for $i \neq j$ and $q > 1$. A similar relation holds true for the function $\tilde{f}(z)$ given by

$$\tilde{f}(z) = \frac{f_j(z)}{f_i(z)},$$

it being assumed that $\tilde{f}(z)$ is an entire function.

Proof Since the result is obvious when

$$\sigma_{f_i, f_j}(p, q) = 0,$$

we suppose that

$$\sigma_{f_i, f_j}(p, q) > 0.$$

We can clearly assume that $\sigma_{f_k}(p_k, q_k)$ ($k = i, j$) is finite. We assume also that any one of the first four conditions of the Proposition of Section 2 is satisfied for $i \neq j$.

Let

$$\max \{ \rho_{f_i}(p_i, q_i), \rho_{f_j}(p_j, q_j) \} = \rho_{f_i}(p_i, q_i) = \rho$$

and

$$\sigma_{f_i}(p_i, q_i) = \sigma.$$

We further let $\varepsilon > 0$ and $\varepsilon_1 > 0$ be arbitrary.

We begin by considering the following expression:

$$\frac{\exp^{[p-2]}(\sigma + \varepsilon) \left(\log^{[q-1]} r \right)^\rho}{\exp^{[p-2]} \left(\sigma + \frac{\varepsilon}{2} \right) \left(\log^{[q-1]} r \right)^\rho}$$

for all sufficiently large values of r . Indeed, for any $\delta > 1$, it follows from the above expression, for all sufficiently large values of $r \geq r_1 \geq r_0$, that

$$\frac{\exp^{[p-2]}(\sigma + \varepsilon) \left(\log^{[q-1]} r_0 \right)^\rho}{\exp^{[p-2]} \left(\sigma + \frac{\varepsilon}{2} \right) \left(\log^{[q-1]} r_0 \right)^\rho} = \delta \quad (\delta > 1). \quad (4.42)$$

Now, in view of (1.5), we find from () that, for all sufficiently large values of r ,

$$\begin{aligned} M_{f_i, f_j}(r) &\leq \exp^{[p_i-1]} \left[\left(\sigma_{f_i}(p_i, q_i) + \frac{\varepsilon}{2} \right) \left(\log^{[q_i-1]} r \right)^{\rho_{f_i}(p_i, q_i)} \right] \\ &\quad \times \exp^{[p_j-1]} \left[\left(\sigma_{f_j}(p_j, q_j) + \frac{\varepsilon}{2} \right) \left(\log^{[q_j-1]} r \right)^{\rho_{f_j}(p_j, q_j)} \right], \end{aligned}$$

that is,

$$M_{f_i, f_j}(r) \leq \exp^{[p-1]} \left[\left(\sigma + \frac{\varepsilon}{2} \right) \left(\log^{[q-1]} r \right)^\rho \right] \\ \times \exp^{[p_j-1]} \left[\left(\sigma_{f_j}(p_j, q_j) + \frac{\varepsilon}{2} \right) \left(\log^{[q_j-1]} r \right)^{\rho_{f_j}(p_j, q_j)} \right].$$

Now, in view of any one of the the first four conditions of the Proposition of Section 2 for $i \neq j$, we find for all sufficiently large values of r that

$$\exp^{[p-1]} \left[\left(\sigma + \frac{\varepsilon}{2} \right) \left(\log^{[q-1]} r \right)^\rho \right] \\ > \exp^{[p_j-1]} \left[\left(\sigma_{f_j}(p_j, q_j) + \frac{\varepsilon}{2} \right) \left(\log^{[q_j-1]} r \right)^{\rho_{f_j}(p_j, q_j)} \right]. \quad (4.43)$$

Therefore, it follows from (4.43) that, for all sufficiently large values of r ,

$$M_{f_i, f_j}(r) \leq \exp^{[p-1]} \left[\left(\sigma + \frac{\varepsilon}{2} \right) \left(\log^{[q-1]} r \right)^\rho \right]^2,$$

that is,

$$M_{f_i, f_j}(r) \leq \exp^{[p-1]} \left[(\sigma + \varepsilon) \left(\log^{[q-1]} r \right)^\rho \right].$$

By applying Theorem 4, we get from the above observations that, for all sufficiently large values of r ,

$$\frac{\log^{[p-1]} M_{f_i, f_j}(r)}{\left(\log^{[q-1]} r \right)^\rho} < \sigma + \varepsilon,$$

that is,

$$\limsup_{r \rightarrow \infty} \left\{ \frac{\log^{[p-1]} M_{f_i, f_j}(r)}{\left(\log^{[q-1]} r \right)^{\rho_{f_i, f_j}(p, q)}} \right\} \leq \sigma + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\sigma_{f_i, f_j}(p, q) \leq \sigma_{f_i}(p_i, q_i). \quad (4.44)$$

Next, without any loss of generality, we may assume that

$$f_k(0) = 1 \quad (k = i, j).$$

Also let $r > R$. Then, we find from (4.17), for a sequence of values of R tending to infinity, that

$$M_{f_i}(R) \geq \exp^{[p-1]} \left[\left(\sigma_{f_i}(p_i, q_i) - \varepsilon \right) \left(\log^{[q-1]} R \right)^\rho \right]. \quad (4.45)$$

Furthermore, by using (4.16), we have for all sufficiently large values of r that

$$M_{f_j}(r) \leq \exp^{[p_j-1]} \left[\left(\sigma_{f_j}(p_j, q_j) + \varepsilon \right) \left(\log^{[q_j-1]} r \right)^{\rho_{f_j}(p_j, q_j)} \right].$$

Since, in view of any one of the first four conditions of the Proposition of Section 2, we have

$$\exp^{[p_j-1]} \left[\left(\sigma_{f_j}(p_j, q_j) + \varepsilon \right) \left(\log^{[q_j-1]} r \right)^{\rho_{f_j}(p_j, q_j)} \right] \\ < \exp^{[p-1]} \left[(\sigma + \varepsilon) \left(\log^{[q-1]} r \right)^\rho \right],$$

we readily conclude that

$$M_{f_j}(r) < \exp^{[p-1]} \left[(\sigma + \varepsilon) \left(\log^{[q-1]} r \right)^\rho \right]. \quad (4.46)$$

Since $r > R$, we find from (4.45), for a sequence of values of r tending to infinity, that

$$M_{f_i}(r) > M_{f_i}(R) > \exp^{[p-1]} \left[(\sigma - \varepsilon) \left(\log^{[q-1]} R \right)^\rho \right] \\ > \exp^{[p-1]} \left[(\sigma - \varepsilon) \left(\log^{[q-1]} \frac{r}{2} \right)^\rho \right]. \quad (4.47)$$

Suppose now that z_r is a point on the circle $|z| = r$ such that

$$M_{f_i}(r) = |f_i(z_r)|.$$

Then, since $r > R$, it follows from (4.37), (4.46) and (4.47) that, for a sequence of values of r tending to infinity,

$$M_{f_i, f_j}(r) \geq \left(\exp^{[p-1]} \left[(\sigma + \varepsilon) \left(\log^{[q-1]} 4eR \right)^\rho \right] \right)^{-7} \\ \times \exp^{[p-1]} \left[(\sigma - \varepsilon) \left(\log^{[q-1]} \frac{r}{2} \right)^\rho \right],$$

that is,

$$M_{f_i, f_j}(r) \geq \left(\exp^{[p-1]} \left[(\sigma + \varepsilon) \left(\log^{[q-1]} 4er \right)^\rho \right] \right)^{-7} \\ \times \exp^{[p-1]} \left[(\sigma - \varepsilon) \left(\log^{[q-1]} \frac{4er}{8e} \right)^\rho \right]. \quad (4.48)$$

We also have

$$\lim_{r \rightarrow \infty} \left\{ \frac{\exp^{[p-2]} \left[(\sigma - \varepsilon) \left(\log^{[q-1]} \frac{4er}{8e} \right)^\rho \right]}{\exp^{[p-2]} \left\{ (\sigma + \varepsilon) \left(\log^{[q-1]} 4er \right)^\rho \right\}} \right\} = \infty.$$

So, for all sufficiently large values of r with $r_n > r_1 > r_0$, we may write

$$\frac{\log \left[(\sigma - \varepsilon) \left(\log^{[q-1]} \frac{4er_n}{8e} \right)^\rho \right]}{\log \left[(\sigma + \varepsilon) \left(\log^{[q-1]} 4er_n \right)^\rho \right]} > \frac{\log \left[(\sigma - \varepsilon) \left(\log^{[q-1]} \frac{4er_0}{8e} \right)^\rho \right]}{\log \left[(\sigma + \varepsilon) \left(\log^{[q-1]} 4er_0 \right)^\rho \right]} =: \delta.$$

Therefore, clearly, we obtain

$$\delta > 1.$$

Consequently, for the above value of δ , it can easily be verified that

$$\exp^{[p-1]} \left[(\sigma - \varepsilon) \left(\log^{[q-1]} \frac{4er}{8e} \right)^\rho \right] \\ \geq \exp^{[p-1]} \left[\left\{ (\sigma + \varepsilon) \left(\log^{[q-1]} 4er \right)^\rho \right\}^\delta \right]. \quad (4.49)$$

Also, if we apply Lemma 2, we find for all sufficiently large values of r that

$$\begin{aligned} & \exp^{[p-1]} \left[\left\{ (\sigma + \varepsilon) (\log^{[q-1]} 4er)^\rho \right\}^\delta \right] \\ & \geq \left(\exp^{[p-1]} \left[(\sigma + \varepsilon) (\log^{[q-1]} 4er)^\rho \right] \right)^8. \end{aligned} \quad (4.50)$$

Now, in light of Theorem 4, it follows from (4.48), (4.49) and (4.50) that, for all sufficiently large values of r ,

$$M_{f_i, f_j}(r) \geq \exp^{[p-1]} \left[(\sigma + \varepsilon) (\log^{[q-1]} 4er)^\rho \right], \quad (4.51)$$

that is,

$$\frac{\log^{[p-1]} M_{f_i, f_j}(r)}{(\log^{[q-1]} 4er)^\rho} \geq (\sigma + \varepsilon),$$

that is,

$$\limsup_{r \rightarrow \infty} \left\{ \frac{\log^{[p-1]} M_{f_i, f_j}(r)}{(\log^{[q-1]} r + O(1))^{\rho_{f_i, f_j}(p, q)}} \right\} \geq \sigma + \varepsilon \quad (q > 1), \quad (4.52)$$

so that

$$\sigma_{f_i, f_j}(p, q) \geq \max \left\{ \sigma_{f_i}(p_i, q_i), \sigma_{f_j}(p_j, q_j) \right\} \quad (q > 1). \quad (4.53)$$

So, clearly, the first part of Theorem 5 follows from (4.44) and (4.53).

The part of the proof for the function $\tilde{f}(z)$ given by

$$\tilde{f}(z) = \frac{f_j(z)}{f_i(z)}$$

can easily be carried out along the lines of the corresponding part of the proof of Theorem 4. Therefore, we omit the details involved.

The proof of Theorem 5 is thus completed. $\square$

Our next result (Theorem 6) provides the condition under which the equality sign in the assertion (4.28) of Theorem 4 holds true in the case of the condition (v) of the Proposition of Section 2.

THEOREM 8 Let $f_1(z)$ and $f_2(z)$ be any two entire functions such that

$$\rho_{f_1}(p, q) = \rho_{f_2}(p, q) \quad (0 < \rho_{f_1}(p, q) = \rho_{f_2}(p, q) < \infty)$$

and

$$\sigma_{f_1}(p, q) \neq \sigma_{f_2}(p, q).$$

Then

$$\rho_{f_1, f_2}(p, q) = \rho_{f_1}(p, q) = \rho_{f_2}(p, q) \quad (p, q \in \mathbb{N}; p \geq q > 1). \quad (4.54)$$

Proof The proof of Theorem 6 is much akin to that of Theorem 3, so we choose to omit the details involved. $\square$

5. Conclusion

In Theorem 1, Theorem 2, Theorem 4 and Theorem 5 of our present investigation, we have discussed about the limiting value of the lower bound under any one of the first four conditions of the Proposition of Section 2. Moreover, in Theorem 3 and Theorem 6, we have also determined the limiting value of the lower bound in Case (v) of the Proposition under some significantly different conditions. Naturally, therefore, a question may arise about the limiting value of the lower bound when any one of the last five cases of the Proposition is considered. This may provide scope for study for the interested future researchers in this subject.

Funding

The authors received no direct funding for this research.

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Citation information

Cite this article as: Sum and product theorems depending on the (p, q) -th order and (p, q) -th type of entire functions, H.M. Srivastava, Sanjib Kumar Datta, Tanmay Biswas & Debasmita Dutta, *Cogent Mathematics* (2015), 2: 1107951.

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